MAT213a: Homework 6 – Solutions

1. Recall the identity

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\Big(\frac{1}{z-n} + \frac{1}{z+n}\Big) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}.$$

   Our statement is the special case when $z = i$.

2. To show that $\theta(z)$ is anaytic, we simply check that it converges absolutely and uniformly on compact sets.

$$|\theta(z)| \le \sum |e(n^2\tau + nz)| = \sum e\Big(-n^2\cdot\mathrm{Im}(\tau) + n\cdot\mathrm{Im}(z)\Big).$$

   Since $n^2 \gg n$, we see that the series for theta is bounded by the exponential series. The fact that θ is order 2 follows from the periodicity: $\theta(z+1) = \theta(z)$ and $\theta(z+2\tau) = e^{-2\pi i(\tau+z)}\theta(z)$ (the growth comes from the τ-direction).

3. By Gauss' formula,

$$\begin{aligned}
\pi/2 = \Gamma(1/2)^2/2 &= \lim_{n\to\infty}\frac{1}{2}\left(\frac{n!\cdot n^{1/2}}{1/2\cdot 3/2\cdots(n+1/2)}\right)^2\\
&= \lim_{n\to\infty}\frac{4n}{2}\left(\frac{1\cdot 2\cdots n}{3/2\cdots(n+1/2)}\right)^2\\
&= \lim_{n\to\infty} 2n\cdot\frac{2\cdot 2\cdot 4\cdot 4\cdots(2n)(2n)}{3\cdot 3\cdots(2n+1)(2n+1)}\\
&= \lim_{n\to\infty}\frac{2\cdot 2\cdot 4\cdot 4\cdots(2n)(2n)}{1\cdot 3\cdot 3\cdots(2n-1)(2n+1)}.
\end{aligned}$$

4. It is obvious that $f(z+1) = 2f(z)$ if $g(z) = f(z)/2^z$ is periodic, i.e $g(z+1) = g(z)$. Thus $g(z) = G(e^{2\pi iz})$ for some analytic function on $\mathbb{C}^*$. Thus $f(z) = 2^zG(e^{2\pi iz})$. Conversely, such functions satisfy the required property. Examples: $2^z\cdot e^{2\pi izk}$ have finite order.

5. To check that $F(s)$ and $\Gamma(s)$ are related, it might be useful to compute $F(s+1)$:

$$\begin{aligned} F(s+1) &= \int_0^1 x^{s+1}(1-x)^a \frac{dx}{x(1-x)} \\ &= \frac{s}{a}\int_0^1 x^s(1-x)^{a+1}\frac{dx}{x(1-x)} \\ &= \frac{s}{a}\int_0^1 x^s(1-x)^a(x+(1-x))\frac{dx}{x(1-x)} \\ &= \frac{s}{a}F(s) - \frac{s}{a}F(s+1). \end{aligned}$$

Thus $F(s+1) = sF(s)/(s+a)$. This is very cool, looks very much like the Γ function but there's an extra factor of $(s+a)$. To get rid of it, we define $G(s) = F(s)\Gamma(s+a)$ so that $G(s+1) = sG(s)$. Do $G(s)$ and $\Gamma(s)$ agree at a point?

$$G(1) = F(1)\Gamma(a+1) = a\Gamma(a)\int_0^1 (1-x)^{a-1}dx = \Gamma(a).$$

Not quite, but is $G(s) = \Gamma(a)\Gamma(s)$? From the integral expression for F, we have $|F(x)| \geq |F(x+iy)|$, so we can appeal to Wielandt's theorem.

6. We substitute $x = t^{1/a}, dx = (1/a)t^{1/a-1}$:

$$\int_0^1 (1-x^a)^b dx = \frac{1}{a}\int_0^1 (1-t)^b t^{1/a}\frac{dt}{t} = \frac{1}{a}\int_0^1 (1-t)^{b+1}t^a \frac{dt}{t(1-t)}.$$

By the previous problem, this is just $\Gamma(b+1)\Gamma(1/a)/\Gamma(b+1+1/a)$.

7. Clearly, $(z-a)$ is prime, as it consists of functions which vanish at a point. Conversely, suppose the principal ideal is generated by (f). If f has no zeros, it is invertible, hence the ideal is everything. If f has a zero at a point a, then $f(z) = (z-a)^k g(z)$ with $g(z) \neq 0$. If $g(z)$ is a unit, then the ideal is just $(z-a)^k$ and it is prime if and only if $k=1$. Finally, if $g(z)$ is not a unit, then $f(z)$ is a product of two functions not divisible by f, hence (f) is not prime.

8. Consider the map $f(z) = \frac{a+b}{2}\cdot z + \frac{a-b}{2}\cdot\frac{1}{z}$. Then it is affine on the circle, and hence sends it to an ellipse. To check injectivity, suppose $f(z) = w$. This means that

$$z^2 - \frac{2w}{a+b}\cdot z + \frac{a-b}{a+b} = 0.$$

When w is large, one root must be inside the unit disk and the other outside because the sum of the roots is large while the product is fixed. By the argument principle, we see the same is true for any $w \in \mathbb{C} \setminus E(a,b)$.

9. The map in problem 8 takes $\mathbb{H} \setminus \overline{\Delta}$ to $\mathbb{H} \setminus E(a,b)$, so we only need to compose it with the inverse of $z + 1/z$ (it is $z \to z + (1/2)\sqrt{z^2 - 4}$) to get a conformal map from the upper half-plane.

10. We first take the disk to the upper half plane by $z \to i \cdot \frac{1+z}{1-z}$. Then we map the upper half plane to $\mathbb{C} \setminus [0, \infty)$ by squaring $z \to z^2$. Finally, we take $\mathbb{C} \setminus [0, \infty)$ to $\mathbb{C} \setminus [-\infty, -1/4)$ by $z \to -z - 1/4$. The composition is

$$z \to \frac{z}{(1-z)^2}.$$

Expanding, we find $z + 2z^2 + 3z^3 + \dots$.