MAT213A: Homework 5 – Solutions

1. Suppose to the contrary that $f(z) = p(z) - e^{\lambda z}$ had only finitely many zeros for some polynomial $p(z)$ which does not vanish identically. As f has order 1, by Hadamard's factorization theorem, we would have $f(z) = q(z)e^{az}$ for some polynomial $q(z)$. But is it really possible that $q(z)e^{az} = p(z) - e^{\lambda z}$? By comparing growth rates, we would have to have $a = \lambda$, $q \equiv 0$, $p \equiv 0$. Contradiction.

2. It is well known that $M(r)$ is a convex function, the assumptions force it to be linear in a non-trivial interval, call the slope k. But then $f(z)z^{-k}$ (locally defined) has constant growth on that interval. By the maximal principle, we see that $f(z)z^{-k}$ is constant, and so $f(z) = az^k$ for some $a \neq 0$. Finally, as f is single-valued, $k = n$ must be a non-negative integer.

3. Let $f(0) = 1$ be a bounded holomorphic function on the unit disk with zeros $\{a_k\}$. We apply Jensen's formula. For $r < 1$, we have
$$0 = \log|f(0)| = \frac{1}{2\pi}\int_{|z|=r} \log|f(re^{i\theta})|d\theta - \sum_{k=1}^{n} \log\frac{r}{|a_k|}$$
Thus by Jensen's inequality(!),
$$\sum_{k=1}^{n} \log\frac{r}{|a_k|} = \frac{1}{2\pi}\int_{|z|=r} \log|f(re^{i\theta})|d\theta \leq \log\frac{1}{2\pi}\int_{|z|=r} |f(re^{i\theta})|d\theta < C.$$
Tending $r \to 1$, we see that $\sum_{k=1}^{\infty} \log|a_k| > -C$. Since we know that $a_k \to 1$, the sum $\sum(1 - |a_k|)$ converges.

4. The cosine function has zeros at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$, so the zeros of $\cos\sqrt{z}$ are $\frac{1}{4}\pi^2, \frac{1}{9}\pi^2, \ldots$. As $\cos\sqrt{z}$ has order $1/2$, Hadamard's factorization theorem tells us that
$$\cos\sqrt{z} = C\prod_{k=0}^{\infty}\Big(1 - \frac{4z}{(2k+1^2\pi^2)}\Big).$$
By plugging $z = 0$, we find that $C = 1$.

5. Recall that the order of a canonical product with zeros at $\{a_n\}$ is the smallest integer p for which the sum $\sum 1/|a_n|^{p+1}$ converges. Thus to achieve a canonical product of order $\rho = 1$, we could use $a_n = n(\log n)^2$.

6. The functions $f(z), f(z+1), f(z+i)$ have the same order, and the same zeros, so by Hadamard's factorization theorem, there exist polynomials P, Q for which $f(z+1) = e^{P(z)} f(z)$ and $f(z+i) = e^{Q(z)} f(z)$. For the second part, if $P = Q \equiv 0$, then $f(z)$ would be doubly-periodic, would have a compact fundamental domain, and therefore, constant.

7. The zeros of $\cos(\pi z)$ are the odd integers $\pm 1, \pm 3, \dots$, the zeros of $\cos(\pi z/2)$ are the integers $\pm 2, \pm 6, \dots$ which are once divisible by two. Thus the infinite product

$$\prod_{n=1}^{\infty} \cos(\pi/2^n z) \qquad \text{and} \qquad \sin(\pi z)/(\pi z)$$

have the same zeros. As these are entire functions of order 1, they certainly must agree up to a factor of e^{az+b}, but as these functions are even, $a = 0$, and plugging $z = 0$, we see that $b = 0$ as well. Our problem is the special case of the above identity with $z = \pi/2$.

8. Using the canonical product for $1/\Gamma$, it is easy to see that it has order 1. We now study the behaviour of $1/\Gamma$ in the direction of the negative real ray. Since $\Gamma(z) = z\Gamma(z)$ and $\Gamma(1/2) = \sqrt{\pi}$, we find that $|1/\Gamma(-n+1/2)| \gtrsim n!$. This prevents us from bounding $M_r(1/\Gamma(z))$ by e^{Cr}.

9. As $1/\Gamma(3z)$ has the same zeros as $1/(\Gamma(z)\Gamma(z+1/3)\Gamma(z+2/3))$, we have

$$\Gamma(z)\Gamma(z+1/3)\Gamma(z+2/3) = e^{az+b}\Gamma(3z). \qquad (*)$$

We must evaluate a, b. Plugging in $z = 1/3$ into the formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, we find that $\Gamma(1/3)\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}$. Also note that $\Gamma(4/3) = 1/3 \cdot \Gamma(1/3)$.

Now we plug $z = 1/3$ and $z = 2/3$ into $(*)$:

$$\begin{aligned} \log(2\pi/\sqrt{3}) &= a/3 + b \\ -\log 3 + \log(2\pi/\sqrt{3}) &= 2a/3 + b \end{aligned}$$

Solving, we find $a = -3\log 3$ and $b = \log(2\pi) + (\log 3)/2$ which tells us that

$$\Gamma(z)\Gamma(z+1/3)\Gamma(z+2/3) = 2\pi 3^{1/2-3z}\Gamma(3z).$$

10. Recall the duplication formula: $2\sqrt{\pi}\Gamma(2t) = 2^{2t}\Gamma(z)\Gamma(t+1/2)$. Let $I = \int_0^1 \log\Gamma(t)dt$. We have

$$\begin{aligned} I &= 2\int_0^{1/2} \log\Gamma(2t)dt \\ &= 2\int_0^{1/2} \log\Big(2^{2t}\Gamma(t)\Gamma(t+1/2)\cdot\frac{1}{2\sqrt{\pi}}\Big)dt \\ &= 2\Big(\int_0^{1/2} 2t\log 2dt + \int_0^1 \log\Gamma(t)dt - \int_0^{1/2}\log 2\sqrt{\pi}dt\Big) \\ &= \frac{1}{2}\log 2 + 2I - \log 2\sqrt{\pi}. \end{aligned}$$

Hence $I = \log\sqrt{2\pi}$.