MAT213A: Homework 4 – Solutions

1. No, maps like $z \to z + i$ and $z \to z - 1/z$ are not automorphisms but still do not contract strictly. To check that the first map does not contract strictly, notice that

$$d(iy, i(y+1)) = \log \frac{y+1}{y}.$$

Thus

$$\frac{d(i(y+1), i(y+2))}{d(iy, i(y+1))} = \log \frac{y+2}{y+1} \div \log \frac{y+1}{y} \to 1.$$

2. A covering map from the upper half-plane to the punctured disk is $w(z) = e^{iz}$. Hence, $z = \frac{\log(w)}{i}$. We compute the hyperbolic metric of the punctured disk:

$$\frac{|dz|}{\text{Im } z} = \frac{|dw|}{|w| \cdot \text{Im}(\log(w)/i)}.$$

Hence $\rho(w) = -\frac{1}{|w| \log |w|}$, whose square is well known to be integrable near the origin.

3. The covering map from $\mathbb{H} \to A(1, R)$ is

$$z \to \exp\Big(\frac{1}{\pi i} \log(R) \log(z)\Big)$$

After some computation, one finds that

$$\rho(z) = \frac{\pi}{|z| \log R \cdot \sin\Big(\pi \cdot \frac{\log |z|}{\log R}\Big)}.$$

4. We claim that $\frac{|dw|}{\sqrt{4-w^2}}$ is the desired metric. Indeed, if $z \in S^1$ and $w(z) = z + 1/z$,

$$\frac{|dw|}{\sqrt{4 - w(z)^2}} = \frac{(1 - 1/z^2)|dz|}{\sqrt{4 - 4(\text{Re } z)^2}} = \frac{(z - 1/z)|dz|}{2|\text{Im } z|} = |dz|.$$

Orthogonality follows from the fact that z^m is orthogonal to z^n on the unit circle for $m \neq n$.

5. Suppose f is a proper map from $\mathbb{H} \to \mathbb{H}$. If we conjugate f by a Möbius transformation, we get a proper map $\tilde{f}$ from $\Delta \to \Delta$. Since $\tilde{f}$ admits a Schwarz reflection, $\tilde{f}$ extends continuously to $\partial\Delta$. Here we can conjugate Blaschke products over to the

upper half-plane, but I will give a solution based on the upper half-plane. (The point of this paragraph is to explain that $\infty \subset \mathbb{R} \cup \{\infty\}$ is not a special point.)

We can conclude that f has finitely many poles at $b_1, \ldots, b_d$ on the real axis and possibly a pole at infinity. Furthermore, since $f : \mathbb{H} \to \mathbb{H}$, near a finite pole, f must look like $-a_i/(z - b_i)$ (and the pole must be simple) for some $a_i \geq 0$. Similarly, the possibility at infinity is $a_0 z$. As f is proper, there are no poles in $\mathbb{H}$ and by symmetry, there are no poles in $\overline{\mathbb{H}}$. By subtracting off the poles, with get a poleless rational function, which must be constant. Thus any proper map is of the desired form. Conversely, all maps of the form $f(z) = a_0 z + b + \sum_1^d -a_i/(z - b_i)$ are sums of functions which map the upper half-plane to itself and preserve $\mathbb{R} \cup \{\infty\}$ and so are proper.

6. I only explain the more difficult assertions. A is an integral domain because the domain is connected and zero sets of analytic functions are discrete. A is integrally closed in K: if not, then we would have an equation $f^n + a_{n-1} f^{n-1} g + \cdots + a_n g^n \equiv 0$ for some $f/g \in K \setminus A$. But since the quotient is non-trivial, g has to vanish at some point where f does not vanish, which is incompatible with the equation. Finally K is not algebraically closed, for example $f(z) = z$ does not admit a square root.

7. The fact that $A(\rho) \subset A$ forms a subring is obvious. If it is not integrally closed in K, then by the previous problem, we would have an equation $f^n + a_{n-1} f^{n-1} + \cdots + f_0$ with $f \in A \setminus A(\rho) \equiv 0$ which is impossible, since f^n dominates the other terms on a sequence of points tending to infinity.

8. By Cauchy's estimate, $|f'(z)| \lesssim M(2R)/R$ and these additional factors do not affect the growth order.

9. This problem is a consequence of the well known formula for order ρ of an entire function $\sum a_n z^n$ in terms of the size of the coefficients:

$$\rho = \limsup_{n \to \infty} \frac{n \log n}{-\log |a_n|}$$

Sketch of proof: a bound on the rate of growth, yields a bound on the coefficients by Cauchy's estimates. The converse is established by using Stirling's formula to estimate the series. By this formula, the order of the function in question is $1/\alpha$.

10. We claim $\alpha = \alpha'$. By the definition of α', there exists arbitrary large r for which $N(r) > r^{\alpha'-\epsilon}$ (is LARGE). Then for any $\beta < \alpha' - \epsilon$, if we drop zeros outside $|z| = r$, we see that

$$\sum_m |a_m|^{-\beta} \geq r^{-\beta} N(r) \to \infty.$$

It follows that the critical exponent is at least α'. Conversely, since $N(r) < r^{\alpha'+\epsilon}$ (is SMALL) for arbitrary large r. Grouping the zeros dyadically, i.e by $|a_m| \in [2^n, 2^{n+1}]$ we see that for $\beta > \alpha' + \epsilon$,

$$\sum_m |a_m|^{-\beta} \lesssim \sum_n 2^{-n\beta} N(2^n) < \infty$$

as the tails are dominated by a geometric series. Hence, the critical exponent also does not exceed α', and thus must equal α'.