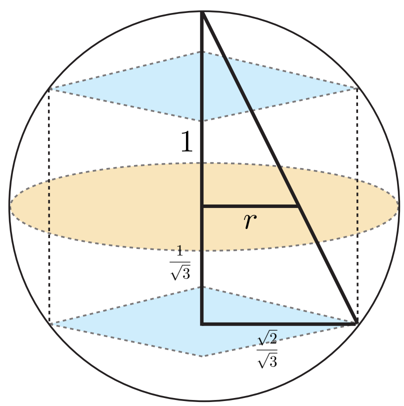
MAT213a: Homework 3 – Solutions

1. A $\mathbb{C}$–linear automorphism of $\mathbb{C}(z)$ is determined by the image of the identity map $f(z) = z$. As the image should be invertible, it can be mapped to any invertible element, so the Galois group is $\mathrm{PSL}(2, \mathbb{Z})$.

2. There should be two solutions for r, one with $r < 1$ and one with $r > 1$ (they are reciprocals). Let us find the solution with $r < 1$. A spherical square with angles $2\pi/3$ has area $2\pi/3$, which is 1/6-th of the total surface area, so it comes from the cube.



From the diagram, we compute

$$\frac{r}{1} = \frac{\sqrt{2}/\sqrt{3}}{1 + 1/\sqrt{3}}.$$

and hence $r = \frac{\sqrt{2}}{1+\sqrt{3}}$. The other solution is of course $r = \frac{1+\sqrt{3}}{\sqrt{2}}$.

3. The spherical geodesics happen to be circles, so under the stereographic projection π, they will be circles (or lines) in the plane. Which ones? Since spherical geodesics are *great* circles, they cut the equator γ in two diametrically opposite points, so they correspond to circles (or lines) in the plane which cut $\pi(\gamma)$ in two diametrically opposite points.

4. This problem is a direct consequence of interpreting the geometry of the Riemann sphere as follows: If we think of the Riemann sphere as $\mathbb{CP}^1$, and $z, w \in \mathbb{CP}^1$, then $\cos d(z, w) = \langle Z, W \rangle$ where Z, W are unit lifts of z, w to $\mathbb{C}^2$.

5. Suppose $v(z) = \sum_{n=0}^{\infty} a_n z^n \frac{\partial}{\partial z}$ (coordinates at the origin) is holomorphic vector field on the Riemann sphere. Switching coordinates to infinity: $z \to 1/w$, $\frac{\partial}{\partial z} \to w^2 \frac{\partial}{\partial w}$. As v is holomorphic, the power series expansion in w cannot have negative terms, so we see that $v(z) = (a_0 + a_1 z + a_2 z^2)\frac{\partial}{\partial z}$ and so the space of holomorphic vector fields is a three dimensional vector space. Now we compute the Lie algebra structure (the commutators):

$$\left[z\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right] = z\frac{\partial}{\partial z}\frac{\partial}{\partial z} - \frac{\partial}{\partial z}\left(z\frac{\partial}{\partial z}\right) = -\frac{\partial}{\partial z},$$

$$\left[z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}\right] = (2z^2 - z^2)\frac{\partial}{\partial z} = z^2\frac{\partial}{\partial z}, \qquad \left[\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}\right] = 2z\frac{\partial}{\partial z}.$$

Set $h = 2\frac{\partial}{\partial z}, e = z^2\frac{\partial}{\partial z}, f = \frac{\partial}{\partial z}$. The above implies that

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

It is well known that these are the commutation relations for sl_2 (for the standard basis vectors).

6. By the Casorati-Weierstrass theorem, a holomorphic automorphism of $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ cannot have essential singularities at 0, 1 or ∞, so must extend to a rational function. As the degree of that rational function is necessarily 1, it must be a fractional linear transformation. Finally, the points $0, 1, \infty$ must be permuted, and since a fractional linear transformation is determined by its values at three points, we see that

$$\mathrm{Aut}(\hat{\mathbb{C}} \setminus \{0, 1, \infty\}) = \mathfrak{S}_3.$$

7. A proper analytic map $f : \Delta \to \mathbb{C}$ must extend continuously to $\partial\Delta$, sending the entire boundary to infinity. Applying the Schwarz reflection principle to $1/f$, we see that the zeros are not isolated. Similarly, a proper map $f : \mathbb{C} \to \mathbb{C}^*$ must extend continuously to a rational function function which omits a point, which is impossible. However, there are proper maps from $\mathbb{C}^* \to \mathbb{C}$ such as $z + 1/z$.

8. Embed the Möbius transformations into $\mathbb{CP}^3$. If a sequence of Möbius transformations does not accumulate to any Möbius transformation, as $\mathbb{CP}^3$ is compact, it must accumulate to some degenerate 2×2 matrix, which has a 1-dimensional kernel (spanned by (z_1, w_1) and 1-dimensional image (spanned by (z_2, w_2)). In this case, we have a subsequence of Möbius transformations tending to the constant function $p = z_1/w_1$ away from the point $q = z_2/w_2$. An example with $p = q = 0$ is given by the sequence of functions $f_n(z) = 1/(nz)$.

9. Let $w(z) = \cos(z)$. What does it mean for $w : (\mathbb{C}, |dz|) \to (\mathbb{C}, \rho(w)|dw|)$ to be a local isometry away from the critical points? It means precisely that

$$\rho(w(z))|w'(z)| = 1.$$

whenever $w'(z) \neq 0$. The critical points of $\cos z$ are πk, on which $\cos z$ attains the values $+1$ and -1 alternatively. Using $\cos^2 z + \sin^2 z = 1$, we see that

$$\rho(w) = \rho(\cos z) = \frac{1}{|\sin z|} = \frac{1}{\sqrt{1 - |\cos^2 z|}} = \frac{1}{\sqrt{1 - |w^2|}}.$$

To visualize the metric $\rho(w)|dw|$, one computes the images of horizontal and vertical lines, which are ellipses and hyperbolas respectively, with foci at ± 1.

10. Since automorphisms of the disk act triple transitively on the boundary, and are isometries in the hyperbolic metric, the areas of all ideal triangles are the same. We compute the area of the ideal triangle shown in the first picture:

$$\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty}\frac{1}{y^2}dydx=\pi.$$

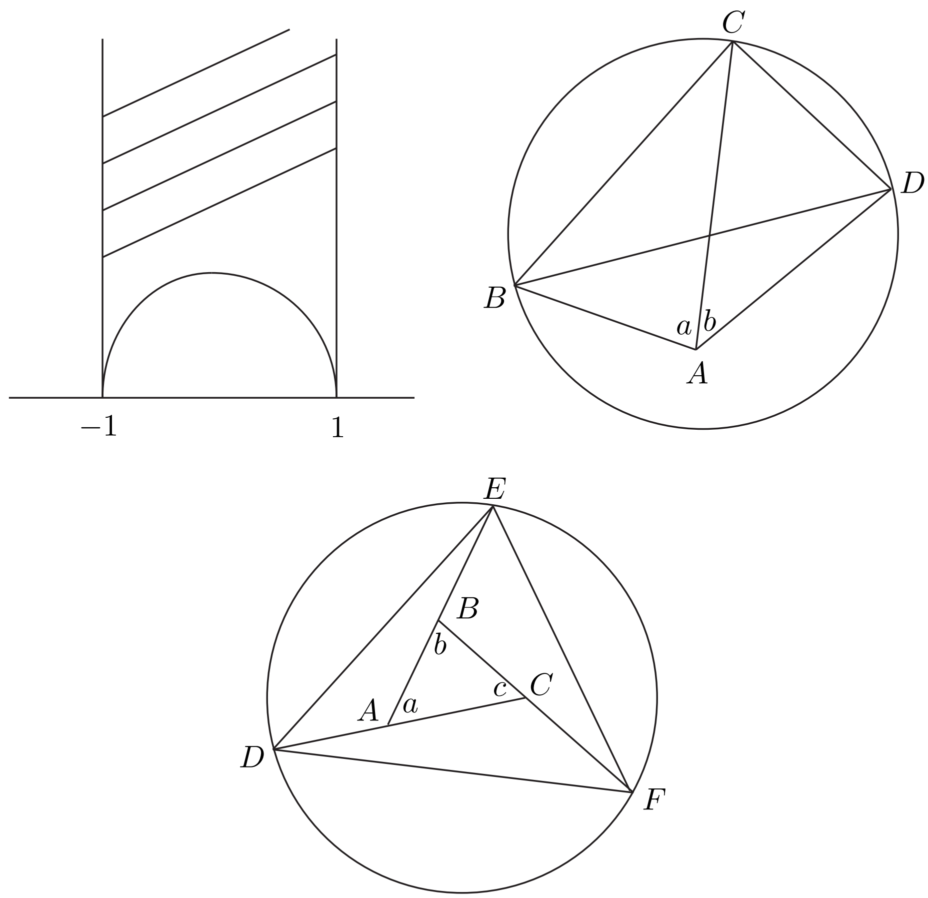


Figure 1: in the schematic diagrams, the Euclidean lines represent hyperbolic lines

From the second picture, we deduce that $T(a,0,0)+T(b,0,0)=T(a+b,0,0)+\pi$, and hence $A(a)+A(b)=T(\pi-a,0,0)+T(\pi-b,0,0)=T(2\pi-a-b,0,0)+\pi=T(\pi,0,0)+T(\pi-a-b,0,0)=A(a+b)$. Since A is an additive and continuous function, it must be linear, so $A(a)=a$.

Finally, from the third picture, one observes that

$$\mathcal{A}(ABC) + \mathcal{A}(ADE) + \mathcal{A}(BEF) + \mathcal{A}(CDF) = \mathcal{A}(DEF) = \pi.$$

Hence, $\mathcal{A}(ABC) = \pi - A(a) - A(b) - A(c) = \pi - a - b - c$.

To find the hyperbolic area of an n-gon, one need only subdivide it into triangles. Adding up the areas of these triangles, one computes that that the area of the polygon is $(n+2)\pi$ minus the sum of the angles.

11. I will only state the theorem and omit the proof, which is a sequence of simple computations.

Theorem: (a) The subgroup $\mathrm{SU}(1,1) \subset \mathrm{SL}(2)$ *preserving the form* $\langle \cdot, \cdot \rangle$ *consists of matrices of form* $A = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$ *which have determinant 1 (up to multiplication by* $\pm I$*). Any such matrix is an automorphism of the unit disk, and conversely all automorphisms of the unit disk can be presented by such matrices. (b) Furthermore, the hyperbolic distance satifies* $\cosh d(z, w) = \langle Z, W \rangle$*.*