MAT213A: Homework 2 – Solutions

1. (i) The fact that $f(z) = \prod_{n=1}^{\infty}(1+q^n)$ is analytic follows from the fact that the product converges uniformly and absolutely on compact subsets of Δ as the sum $\sum q^n$ does. (ii) follows from expanding the product, while (iii) is because f would necessarily have a dense zero set on $\partial\Delta$ as $\lim_{r\to 1} f(re^{2\pi i \cdot p/q}) = 0$ for any rational number p/q. I will prove this explicitly in the case $p/q = -1$ (same exact proof works generally). Let $\delta = 1 - r$. To estimate $|f(-1+\delta)|$, I will group terms $2k+1, 2k+2$ together. The first pair is at most 2δ while other pairs are of type $(1-\alpha)(1+\beta)$ with $\beta < \alpha$, i.e less than 1.

2. We will prove that for any compactly supported smooth function $\phi(z)$, we have

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial\phi/\partial\overline{z}}{z} dz d\overline{z} = \phi(0).$$

Clearly,

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial\phi/\partial\overline{z}}{z} dz d\overline{z} = \frac{1}{2\pi i}\int_{D(0,\epsilon)} \frac{\partial\phi/\partial\overline{z}}{z} dz d\overline{z} + \frac{1}{2\pi i}\int_{\mathbb{C}\setminus D(0,\epsilon)} \frac{\partial\phi/\partial\overline{z}}{z} dz d\overline{z}$$

Now notice that as $1/z$ is an integrable singularity in the complex plane (!), the first term is small when ϵ is small. Now we apply Stokes theorem to the second term:

$$\frac{1}{2\pi i}\int_{\partial D(0,\epsilon)} \frac{\phi(z)}{z} dz \approx \phi(0)\int_{\partial D(0,\epsilon)} \frac{1}{z} dz = \phi(0).$$

(There are two minuses signs coming from the boundary orientation in Stokes theorem and the ordering of $dzd\overline{z}$).

Taking $\epsilon \to 0$, we get the desired result.

3. By the residue theorem, the sum of the residues of the polynomial $1/p$ is 0. Since the degree of p is at least 2, the residue at infinity is 0. It is easy to see that the residue at r_i is $1/p'(r_i)$. Hence $\sum 1/p'(r_i) = 0$.

4. Let $E \subset D(0,R)$. Cover E by compact balls $\{B_i\}_{i=1}^{n}$ with $\sum r_i < \epsilon$. It is clear that

$$f(z) = \int_{|z|=R} \frac{f(\zeta)}{\zeta - z} d\zeta$$

defines a bounded holomorphic function, but is it $f(z$? By Cauchy's integral formula, for $z \in D \setminus E$, we have that

$$f(z) = \int_{|z|=R} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where γ traces the boundaries of the balls. It remains to note that for *specific* z, by taking $\epsilon \to 0$, the second term can be made arbitrarily small.

5. This wonderful sequence $\{L_n\}_{n=1}^{\infty} : 1, 3, 4, 7, 11, 18, 29, 47, \ldots$ satisfies the recurrence $L_{n+2} = L_{n+1} + L_n$ from which it is easy to see that

   $$L_n = \gamma^n + (-1/\gamma)^n$$

   where γ is the golden mean ~ 1.618.

   Let $f(z) = 1 + 2z + 3z^2 + 4z^3 + 7z^4 + \ldots$. Then $f - zf - z^2 f = -z^3 + z + 1$ and so $f(z) = \frac{z^3 - z - 1}{1 - z - z^2}$. (The peculiarity of the first two terms is due to the fact that $|(-1/\gamma)^n| > 1/2$ for $n = 0, 1$ but less than $1/2$ for $n \geq 2$.)

6. The idea is to apply the Baire category theorem to show that on some dense open set, the convergence is uniform. More precisely, let $\{F_M\}$ be the closed sets consisting of points z for which $f_n(z) \leq M$ for all n. By assumptions, $\bigcup_M F_M$ cover the complex plane, so on some open set, there is an m so that F_m has non-empty interior. By Cauchy's bound, we can deduce that the F_m are locally equicontinuous on this open set. By the local nature of the assumptions, we see that arbitrarily close to any point of the complex plane, we have an open set with uniform convergence.

7. Let P be the Poisson kernel, and let $\tilde{P}$ be a harmonic conjugate. Then $f = e^{P + i\tilde{P}}$ is analytic and satisfies the desired property.

8. Given a positive harmonic function u on the unit disk, the analytic map f with real part u takes the unit disk to the right half-plane. Let η be a Mobïus transformation from the right half-plane back to the disk. Then $\eta \circ f$ is bounded, so we can apply Montel's theorem. More precisely, given a sequence of harmonic functions $u_1, u_2, \ldots$, some subsequence of $\eta \circ f_j$'s converges to G, thus this subsequence of u_j's converges to Re $\eta^{-1} \circ G$.

9. Consider the space $L^2(\Delta)$. It is easy to see that the functions $1, z, z^2, \ldots, \overline{z}, \overline{z}^2, \ldots$ are orthogonal. Now Re $z^m = 1/2(z^m + \overline{z}^m)$ while Im $z^m = 1/(2i)(z^m - \overline{z}^m)$, from which it is easy to see that $||\text{Re } z^m|| = ||\text{Im } z^m||$. The desired result follows.

10. Let $D_1, D_2, \ldots$ be a sequence of disjoint disks in the plane tending to infinity, and let g_j be holomorphic functions defined on a neighbourhood of D_j. We will now construct a holomorphic function G which satisfies $|G - g_j| \to 0$ however quickly.

    Let $F_1 \subset F_2 \subset \ldots$ be an exhaustion of the complex plane by compact sets such that F_j contains $D_1, D_2, \ldots, D_{j-1}$ but is disjoint from D_j. By Runge's theorem, there exists a polynomial P_j which is extremely close to 0 on F_j and to $g_j - \sum_{i<j} P_i$ on D_j. Set $G = \sum g_j$.