MAT213A: Homework 13 – Sketches of Solutions

1. The trick is to use that the space of modular forms of weight 8 is 1-dimensional, so one needs to compare the values at $i\infty$ using $G_k(i\infty) = 2\zeta(2k)$.

2. There is only one orbit of $\mathbb{Q} \cup \{\infty\} \subset \partial\mathbb{H}$ when acted by $\mathrm{SL}(2,\mathbb{Z})$. This is seen by the Euclidean algorithm. The generators of $\mathrm{SL}(2,\mathbb{Z})$ are $z \to z+1$ and $z \to -1/z$. Given a fraction m/n, one can use $z \to z+1$ to get rid of the integer part, i.e to make $|m| < n$, and then use $-1/z$ to flip the numerator and denominator. Repeat until you are left with $0/1$.

   For the action of $\Gamma(2)$, you can use a similar procedure to reduce to $0/1, 1/1, 1/0$ depending on whether m/n is even/odd, odd/odd, odd/even respectively. One can easily check that the two standard generators of $\Gamma(2)$ are $z \to z+2, -1/(2z+1)$ preserve the three equivalence classes.

3. M_k is generated as a ring by $dz/z = \eta_{0,\infty}, dz/(z-1) = \eta_{1,\infty}$, and so $\dim M_k = k+1$. One labels dz/z as $\eta_{0,\infty}$ since it has simple poles at 0 and infinity. This object is unique up to a multiplicative constant, as by the Residue theorem, the poles must add up to 0. It is clear that $\eta_{0,\infty} - \eta_{1,\infty} = \eta_{0,1}$. This settles M_1. For M_k, $k \geq 1$, we note that forms have $2k$ poles more than zeros, so there are at least $2k$ poles amongst the locations $0, 1, \infty$ and thus we can divide out by one of the forms $\eta_{0,\infty}, \eta_{1,\infty}, \eta_{0,1}$. Thus we showed that $\eta_{0,\infty}, \eta_{1,\infty}$ generate as a ring, and by looking at the orders of poles, one can see that they cannot have relations.

4. For the square lattice, the critical values of $\wp$ at $0, c_1, c_3, c_2$ respectively are $\infty, e_1, 0, -e_1$. Thus one can post-compose with a Möbius transformation to get a degree 2 cover from the square torus to the sphere with critical values $\omega, \omega^3, \omega^5, \omega^7$ (here $\omega = e^{2\pi i/8}$) as the two four-tuples have the same cross ratio.

   This function shall be called f_i, which we will think as a periodic function on $\mathbb{C}$ with periods 1 and i. Now consider $f_{i/2}(z) = 1/2(f_i(z) + f_i(z)^{-1})$. This function has periods 1 and i, but even better it has the period $i/2$, and so it gives a degree 2 cover from the $i/2$ torus to the Riemann sphere. To see this, one notes that f_i is a Riemann map from the square $(0, 1/2, 1/2 + i/2, i/2)$ to the unit disk, and from the square $(1/2, 1/2 + i/2, 1/2 + i, 1)$ to the exterior of the unit disk. Hence the zeros of f are at $Z_1 = 1/4 + i/4, Z_2 = 3/4 + 3/4 \cdot i$ while the poles are at $P_1 = 1/4 + 3/4 \cdot i, P_2 = 3/4 + 1/4 \cdot i$. Thus the product of f_i and $f_{i+1/2}$ cancels out the zeros and poles, so is constant, moreover this constant is $\omega \cdot \omega^7 = 1$.

The lower "quarter" of the fundamental parallelogram of $\mathbb{Z}[i/2]$ is $(0, 1/2, 1/2 + i/4, i/4)$. FACT: The critical points of a composition $p \circ q$ are the union of the (pre-images) of the ones of p and the ones of q. The points 0 and 1/2 are the original critical points of f_i, while the critical points of $Z \to 1/2(Z + 1/Z)$ are roots of $1 + 1/Z^2$ so ± 1 with critical values also ± 1, and f_i attains the values 1 and -1 at $1/2 + i/4$ and $i/4$ respectively. Thus the answer is the cross-ratio of $-1, -\sqrt{1/2}, \sqrt{1/2}, 1$ in that order.

5. One simply needs to draw (a lift of) the geodesic on the upper-half plane. As a fundamental domain of $\mathbb{H}/\Gamma(2)$, pick the usual one bounded by lines $\text{Re } z = -1, \text{Re } z = 1$ and two circles of radii 1/2 centered at $\pm 1/2$. Then the lift of the geodesic is $\gamma = \gamma_1 \cup \gamma_0$ where γ_1 is the circular arc passing through $-1/2 + 1/2 \cdot i, i, 1/2 + 1/2 \cdot i$ and γ_0 be its translation by one unit to right. One notices that it is a closed loop, since the end points are glued; and in fact, by using the group elements, one can express it as an arc of a unique geodesic in $\mathbb{H}$, so it is a geodesic. Now, γ intersects itself because the point $-1/2 + 1/2 \cdot i$ is glued with $1/2 + 1/2 \cdot i$. Now, γ_0 is a loop which goes around 0 and separates it from 1, ∞, while similarly γ_1 is a loop which separates 1 from 0 and ∞. So γ is the geodesic we seek. One can now easily compute the Poincarë length of γ by computing the length of γ_1 and multiplying the result by 2.

6. (i) This can be done by constructing an infinite sum

$$\sum (z/n)^{2N_n}$$

where N_n are chosen recursively, large enough so that (1) for z real with $|z| \leq n-1$, the term is very small say less than $1/2^n$ guaranteeing the sum converges, (2) dominate M.

(ii) One needs only to apply the first part to $1/(M+1)$.

(iii) Since $z/f(z)$ which is a bounded analytic function, it is constant. This follows that $f(z) = cz$ but then the condition is not satisfied at $z = 0$.

7. It is not true. One gets a grip on $\Gamma'(3/2)$ by either the formula for the logarithmic derivative of Γ or the double-angle formula and then uses numerical computations to show that the expression is not 0.

8. Using $L^2 \implies L^1$ and polar integration, one sees that almost every radial ray has finite length.

9. The complement of the circular arc is first mapped to the complement of a slit, and this is mapped to the complement of the disk.

10. One uses that
$$\alpha/2 = \int_{e_1}^{\infty} \frac{1}{\sqrt{4x^3 - g_2 x - g_3}}$$
and computes the integral for our given lattice.