MAT213A: Homework 12 – Solutions

1. Recall that $(\log\sigma)'' = -\wp(z)$. Thus if we define

$$\zeta(z) = (\log\sigma(z))' = \frac{1}{z} + \sum{}' \Big\{ \frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \Big\},$$

   we easily see that $\zeta'(z) = -\wp(z)$. Uniqueness follows easily from the oddness.

2. First notice that the derivative of $\zeta(z+\lambda_i) - \zeta(z)$ is identically 0. It follows that $\zeta(z+\lambda_i) - \zeta(z) = \eta_i$ for some $\eta_i \in \mathbb{C}$.

   Now consider a fundamental parallelogram F with the origin in its interior. We will now integrate ζ along the ∂F. On one hand, this integral is clearly $-\lambda_1\eta_2 + \lambda_2\eta_1$ because the opposite sides cancel except for the translation factor. On the other hand, by the Residue theorem, this integral equals to $2\pi i \cdot \mathrm{Res}_{z=0}(\zeta(z))$. The residue is clearly 1. Putting all of this together, we find that our determinant in question is $2\pi i$.

3. If we take the logarithmic derivative of $\sigma(z+\lambda_i) = \exp(a_i + b_i z)\sigma(z)$, we see that $\zeta(z+\lambda_i) = b_i + \zeta(z)$. Hence $b_i = \eta_i$. To find a_i, we play off the periodicity and the oddness of σ. More specifically,

$$\sigma(\lambda_i/2) = \exp(a_i - b_i\lambda_i/2)\sigma(-\lambda/2) = -\exp(a_i - b_i\lambda_i/2)\sigma(\lambda/2)$$

   Hence $a_i = \pi i + b_i\lambda_i/2$.

4. From the differential equation $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, one can restore the periods by inverting the elliptic integral.

5. This condition is equivalent to the lattice being generated by two conjugate vectors. Indeed, if this is the case, then g_2, g_3 are real from the sums that define them. Conversely, if g_2, g_3 are real, then by the Theorem 5.14 of the course notes, we see that is generated by both $\int_\gamma \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}$ and $\int_{\overline{\gamma}} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}$ where γ is a loop which encloses two roots of the cubic. But these two are clearly conjugate.

6. It is clear that $\wp'(z)$ and $-\sigma(2z)/\sigma(z)^4$ both have a pole of order 3 at three at the origin with the same principal part. It remains to see that both functions are

doubly-periodic. The only dubious function is $-\sigma(2z)/\sigma(z)^4$. We compute

$$\begin{aligned} -\sigma(2z)/\sigma(z)^4 &= -\frac{\sigma(2z)\exp(b_i\lambda_i/2 + b_i(2z+\lambda_i))\exp(b_i\lambda_i/2 + b_i(2z))}{\sigma(z)^4\exp(b_i\lambda_i/2 + b_i z)^4} \\ &= -\frac{\sigma(2z)\exp(2b_i\lambda_i + 4b_i z)}{\sigma(z)^4\exp(2b_i\lambda_i + 4b_i z)} \\ &= -\frac{\sigma(2z)}{\sigma(z)^4}. \end{aligned}$$

7. Recall from class that for the square lattice $\Lambda = \mathbb{Z}[i]$, the Weierstrass $\wp$ function maps the square S with vertices $(0, 1/2, 1/2 + i/2, i/2)$ to the lower half-plane. The boundary of S is mapped to the real axis. Now consider the diagonal joining 0 and $1/2 + i/2$. On the diagonal, we have $\overline{\wp(z)} = \wp(\overline{z}) = \wp(iz) = -\wp(z)$ (using the symmetry of the lattice and evenness of $\wp$); so it mapped to the negative imaginary ray. Thus if we let the triangle T have vertices $(0, 1/2, 1/2 + i/2)$, then $\wp(z)$ maps either of the 3rd or 4th quadrants; by looking at $p(z) \sim 1/z^2$, it is clear that T is mapped to the third quadrant. Hence $f(z) = -\wp(z/2)^2$ maps it to the upper half-plane.