MAT213A: Homework 10 – Solutions

1. (Solution by Eric Larson) The desired map is

$$f(z) = \int_0^z \sqrt{\sin(w)}dw$$

This can be easily seen by looking at the image of the real axis and looking carefully at the argument of $f(z)$.

2. Suppose that $f_n : \Delta \to \mathbb{C} \setminus \{0, 1\}$ was a sequence of maps which have unbounded derivatives, measured in the appropriate metrics. Pre-composing by a Möbius transformation if necessary, we can assume that $|Df_n(0)| = |f_n'(z)| \cdot \frac{1-|z|^2}{1+|f_n(z)|^2}\Big|_{z=0}$ is unbounded. Next rescale f_n to be defined on disks $\Delta(R_n)$ with the derivative at the origin equal to 1. Then $R_n \to \infty$. A normal families argument constructs an entire function, and by Hurwitz' theorem, it omits the values 0, 1, ∞.

3. The fact that every point in the plane is covered, can be easily seen by a billiard argument. For a triangle where points are covered exactly once, consider the equilateral triangle. For a triangle, which covers points infinitely many times, take any triangle with an irrational angle.

4. As f is nowhere zero, we can write $f = e^g$ for some entire function g. The function g can omit at most one point, and so it takes infinitely many of the values in $2\pi\mathbb{Z}$. Therefore, f sends infinitely many points to 1.

5. Divide both sides by g^n. We have $(f/g)^n + 1 = 1/g^n$. The meromorphic function f/g must omit every n-th root of -1 which contradicts Picard's little theorem for meromorphic functions for $n \geq 3$.

6. The quadratic differential

$$\left(\frac{A}{z^2} + \frac{B}{(z-1)^2} + \frac{C-A-B}{z(z-1)}\right)dz^2$$

satisfies the given conditions. If we had two such quadratic differentials q_1, q_2, their difference would be a quadratic differential with at most simple poles at 0, 1, ∞ are holomorphic otherwise. But every quadratic differential has four more poles than zeros, contradiction.

7. (Solution by Maximilian Butz) Given two spherical triangles with the same angles, there is a Möbius transformation which takes the vertices of one to the vertices of the other. So suppose two spherical triangles T_1, T_2 have the same vertices A, B, C and the same angles, must they be the same? Draw T_1. For each side of T_1, notice whether the side of T_2 joining the corresponding vertices lies inside (to the left) or outside (to the right). Clearly, we cannot have two adjacent arcs which both lie inside or outside. It follows, that $T_1 = T_2$ exactly.