MAT213a: Homework I – Solutions

1. (a) Suppose not. By passing to a subsequence, this would mean that there is a sequence of functions f_n which are zero-free on some disk $D(w, r)$ centered at w; yet their uniform limit has a zero at w. But we know that the number of zeros of a holomorphic function can be counted by an integral:

$$1 \leq \frac{1}{2\pi i} \int_{\partial D(w,r)} \frac{f_n'(z)}{f_n(z)} \to \frac{1}{2\pi i} \int_{\partial D(w,r)} \frac{f'(z)}{f(z)} = 0.$$

We have a contradiction. If we only assume that Ω is open and not connected, we can let $f_n = 1/n$ on one component of Ω, while $f_n = C \neq 0$ on another. This way, the limit function will not be identically 0 on Ω.

2. By the classification of Möbius transformations, up to conjugation, each non-identity Möbius transformation is either a translation of $z \to z + 1$ or a spiralling homothety $z \to \lambda z$. Also notice that a translation has one fixed point, and its iterates have the exact same fixed point (in the case of $z \to z + 1$, it is ∞). Similarly, spiraling homotheties have two fixed points, and the iterates also have the exact same fixed points (in case of $z \to \lambda z$, they are 0 and ∞).

If $f(z) = z + 1$, then $g(z) = z + 1/5$ and so has just 1 solution. If $f(z) = \lambda z$, then $g(z) = \eta z$ where η is any of the five fifth roots of λ.

Finally, if $f(z) = z$ is the identity, $g(z)$ could be $e^{2\pi i/5} z + c$ (amongst other options) for any $c \in \mathbb{C}$.

3. Note that $\tan z$ is the quotient of two entire functions $\frac{\sin z}{\cos z}$. The pole of $\tan z$ closest to the origin is $\pi/2$. From this, it follows that the radius of convergence of $\tan z$ is $\pi/2$.

We need to find an N so that

$$R_N = \sum_{n=N+1}^{\infty} a_n \leq \epsilon = 10^{-1000}.$$

Let S be a square of side length 1.5 centered at 0. By Cauchy's bound,

$$|a_n| \leq \frac{12}{2\pi} \cdot \frac{\max_{\zeta \in \partial S} \tan(\zeta)}{(1.5)^{n+1}} = M.$$

Claim. For $Z = e^{i\zeta}$ with $\zeta \in \partial S$, we have $|Z + 1/Z| > 10^{-4}$.

Assuming the claim, we have

$$|\tan \zeta| \leq \frac{e^{i\zeta} - e^{-i\zeta}}{e^{i\zeta} + e^{-i\zeta}} \leq \frac{2e^{\pi}}{e^{i\zeta} + e^{-i\zeta}} \leq 10^6$$

Thus $R_N \leq 10^{10} \cdot (1.5)^{-N}$. Taking logarithms, we see that $N = 6000$ suffices.

Proof of claim. We need to estimate $|Z^2 + 1|$ from below. On the horizontal sides, the absolute value of $Z^2 = e^{2i\zeta}$ is $e^{\pm 3}$, so

$$|Z^2 + 1| > \min(|e^3 - 1|, |e^{-3} - 1|) > 0.5.$$

On the vertical sides, we use $\arg Z^2 = \pm 3$, so the distance from Z^2 to -1 is at least the projection of -1 to any of these two lines. Using the fact that $e^{3i} \sim -0.989 + 0.141i$, this is easily seen to be at least 0.01.

4. Let $f(x) = \frac{x^6}{(1+x^4)^2}$ denote the integrand; and let γ_R be the following path: a straight line from $-R$ to R along the real axis, and then a semi-circle in the upper-half plane from R to $-R$. The integral over the semi-circle goes to zero as the degree of the denominator is at least 2 greater than that of the numerator. The poles of $f(z)dz$ enclosed by γ are $i^{1/2}$ and $i^{3/2}$.

$$\int_{-\infty}^{\infty} \frac{x^6}{(1+x^4)^2} dx = \lim_{R \to \infty} \int_{\gamma_R} \frac{z^6}{(1+z^4)^2} dz = 2\pi i \sum \text{Res}.$$

Let $W = \{i^{1/2}, i^{3/2}, i^{5/2}, i^{7/2}\}$. By partial fractions,

$$f = \sum_{w \in W} \left\{ \frac{1}{16} \cdot \frac{1}{(x-w)^2} + \frac{3}{16} \cdot \frac{1}{x(x-w)} \right\}$$

The residues come from the second term:

$$2\pi i \sum \text{Res} = 2\pi i \cdot \frac{3}{16} \Big(\frac{1}{i^{1/2}} + \frac{1}{i^{3/2}} \Big) = \frac{3\pi}{4\sqrt{2}}.$$

5. By the maximum principle, $|f|$ achieves its maximum on the boundary. Now if f has no zeros in U, then $1/f$ is a holomorphic function on U, so by the maximal principle again but this time applied to $1/f$, we see that $|f|$ achieves its minimum on the boundary. Thus $|f|$ is constant throughout the domain; and by the mean value property, f must be constant in the domain also.

6. Clearly, we must expand $1/(z-1)$ at infinity and $1/(z-2)$ at 0:

$$\frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1-z/2} = -\frac{1}{2}\Big(1 + \frac{z}{2} + \frac{z^2}{4} + \dots\Big).$$

and

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-1/z} = \frac{1}{z}\Big(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\Big).$$

Using partial fractions, we find

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\Big(\frac{1}{z-2} - \frac{1}{z-1}\Big) =$$

$$= -\frac{1}{z}\Big(\dots + \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \frac{z^3}{16} + \dots\Big).$$

7. As $p(z)$ is holomorphic, the integral

$$\frac{1}{2\pi i} \int_{|z|=1} \Big(p(z) - 1/z\Big) = 1.$$

There must be a point $z \in \{|z| = 1\}$, for which $|p(z) - 1/z|$ is at least the average value, i.e at least 1.

8. Clearly, if f and g are holomorphic, then p is harmonic. Conversely, if p is harmonic, set $f = (1/2)(p + i\tilde{p})$ and $g = (1/2)(p - i\tilde{p})$.

9. Stokes theorem tells us that

$$\int_{\partial U} \overline{z} dz = \int_U d\overline{z} dz = 2i \int_U dx dy,$$

so we can let $f(z) = -(i/2) \cdot \overline{z}$.

10. A covering map from $\mathbb{H} \to \Delta^*$ is $f_H(w) = e^{iw}$. To get a covering map from the disk, one needs to precompose it with the Möbius transformation $w(z) = i \cdot \frac{1+z}{1-z}$.