Complex Analysis: Homework 9

1. Show that the set of Riemann mappings whose images are polygons is dense in S, in the topology of uniform convergence on compact sets.

2. Suppose $f_n \to f$ in S, and $B(p,1) \subset f_n(\Delta)$ for all n. Does it follow that $p \in f(\Delta)$?

3. Suppose $f \in S$ satisfies $f(iz) = if(z)$. Show that $f(\Delta)$ contains $B(0, 1/\sqrt{2})$.

4. Let $U \subset \mathbb{C}$ be a connected open set (which need not be simply-connected), and let $p \in U$. Let $\mathcal{F}$ be the family of all univalent maps $f : U \to \mathbb{C}$ with $f(p) = 0$ and $f'(p) = 1$. Show that $\mathcal{F}$ is compact in the topology of uniform convergence on compact sets.

5. Thinking of the region $U = \mathbb{C} - [1, \infty)$ as a polygon with two vertices, use the Schwarz-Christoffel formula to find the unique Riemann map $f : \Delta \to U$ with $f(0) = 0$ and $f'(0) > 0$.

6. Let $\omega = \omega(z)\, dz$ be a meromorphic 1-form on the Riemann sphere. Prove that $\omega = df$ for some rational function $f(z)$ if and only if $\text{Res}(\omega, p) = 0$ for all $p \in \widehat{\mathbb{C}}$.

7. Given $f : U \to \mathbb{C}$ analytic, let $N(f) = f''(z)/f'(z)\, dz$ as a meromorphic 1-form. (a) Show $N(f) = 0$ iff $f(z) = az + b$ for some a, b. (b) Show $N(f \circ g) = N(g) + g^* N(f)$.

8. Let $f : U \to V$ be a holomorphic map between domains in $\widehat{\mathbb{C}}$. The *Schwarzian derivative* Sf is the quadratic differential defined by $Sf = Sf(z)\, dz^2$ where

$$Sf(z) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 .$$

   Prove that $Sf = 0$ if and only if f is a Möbius transformation, i.e. iff f is the restriction of an element of $\text{Aut}(\widehat{\mathbb{C}})$.

9. Let $\phi(z)$ be an analytic function, and let u_1, u_2 be two linearly independent solutions to the differential equation $d^2u/dz^2 + \phi u = 0$.

   (i) Show $W(z) = \det \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix}$ is a constant.

   (ii) Show that if $f(z) = u_1(z)/u_2(z)$, then $Sf(z) = C\phi(z)$ and find C. Why does the value of Sf not depend on the choice of u_1 and u_2?