Complex Analysis Midterm: Homework 8

All work should be your own. Refer only to class notes (your own and those online) and the course texts.

1. Express $f(z) = \sum_{n=-\infty}^{\infty} (z-n)^{-4}$ in terms of trigonometric functions, and use your result to evaluate $\sum_1^\infty 1/n^4$.

2. State and prove an asymptotic formula for $\Gamma'(x)$ as $x \to +\infty$ along the real axis. (In other words, find a more elementary function $A(x)$ such that $\Gamma'(x)/A(x) \to 1$.)

3. Define $u(z)$ on S^1 by $u(z) = 1$ if $z^2 \in \mathbb{H}$ and $u(z) = 0$ otherwise. Find a formula for the extension of u to a harmonic function on the disk, and draw the locus where $u(z) = 1/2$.

4. Prove that for any $p \geq 1$, the set of analytic functions

$$\mathcal{F} = \left\{ f : \Delta \to \mathbb{C} \ : \ \int |f(z)|^p |dz|^2 \leq 1 \right\}$$

   is compact. That is, every sequence $f_n \in \mathcal{F}$ has a subsequence converging uniformly on compact subsets of Δ, and $f = \lim f_n$ belongs to $\mathcal{F}$ as well.

5. Let $\rho(z)|dz|$ be the hyperbolic metric on a simply-connected domain $U \subset \mathbb{C}$ with $p \in U$, and let $\delta(p) = 1/d(p, \partial U)$ (i) Find (U, p) such that $\rho(p) = 2\delta(p)$. (ii) Find (U, p) such that $\rho(p) = \delta(p)/2$.

6. Is there a univalent map $f : \mathbb{C} - \overline{\Delta} \to \mathbb{C}$ of the form $f(z) = z + \sum_1^\infty b_n/z^n$ with $b_2 = 1/\sqrt{2}$?

7. Let S denote the set of (univalent) Riemann maps $f : \Delta \to \mathbb{C}$ of the form $f(z) = z + \sum_2^\infty a_n z^n$, which the topology of uniform convegence on compact sets. Show that the set of maps whose images are Jordan domains is dense in S.

8. Give an explicit bound $|a_n| \leq M_n$ valid for all $f \in S$. (Use only elementary methods, such as the area theorem or the distortion theorems.)