Complex Analysis: Homework 7

The ring of entire functions $f : \mathbb{C} \to \mathbb{C}$ is denoted $\mathcal{O}(\mathbb{C})$.

1. Prove that for any sequences $a_n, b_n \in \mathbb{C}$ with $a_n \to \infty$, there exists an entire function such that $f(a_n) = b_n$. (Hint: write $f(z) = \sum g_n(z)$ where g_n is a polynomial constructed by induction, such that $g_n(a_i) = 0$ for $i < n$, $g_n(a_n) = b_n - \sum_1^{n-1} g_i(a_n)$, and $|g_n(z)| < 2^{-n}$ when $|z| < |a_n|/2$.)

2. Show that if $f, g \in \mathcal{O}(\mathbb{C})$ have no common zeros, then $(f, g) = (1)$; i.e. show there exist $r, s \in \mathcal{O}(\mathbb{C})$ such that $fr + gs = 1$.

   Hint: write $1/(fg) = F + G$, where F only has poles at the zeros of f and G has zeros only at the zeros of g. (Why is this possible?)

3. Let $I = (f_1, f_2, \ldots) \subset \mathcal{O}(\mathbb{C})$ be the ideal generated by the sequence of functions $f_n(z) = \sin(z/n)/z$. Prove that I is not contained in any proper principal ideal. Conclude that $\mathcal{O}(\mathbb{C})$ is not a PID and that $\mathcal{O}(\mathbb{C})$ contains maximal ideals that are not of the form $(z - a)$.

4. Recall that the *conformal radius* $\rho(U, p)$ is given by $|f'(0)|$ for any holomorphic bijection $f : \Delta \to U$ satisfying $f(0) = p$. Show that if $U \subset V$ then $\rho(U, p) \le \rho(V, p)$.

5. What are the conformal radii of the following pointed regions?

   (a) $(\mathbb{H}, i)$;

   (b) $(\{z : -1 < \operatorname{Re} z < 1\}, 0)$;

   (c) $(\{z \in \mathbb{H} : -\pi < \operatorname{Re} z < \pi\}, i)$;

   (d) $(\widehat{\mathbb{C}} - [-2, 2], \infty)$; and

   (e) (S_α, r), where $r > 0$ and $S_\alpha = \{z : \arg(z) \in (-\alpha, \alpha)\}$.

6. Let $f_n : U \to \mathbb{C}$ be a sequence of analytic functions converging uniformly on compact sets to a nonconstant function $f : U \to \mathbb{C}$. Assume the mappings $f_n(z)$ are at most d-to-1. Show the same is true of $f(z)$.

7. Given $1 \le e < d$, construct a sequence of proper maps $f_n : \Delta \to \Delta$ of degree d such that $f_n \to g$ uniformly on compact sets and $g : \Delta \to \Delta$ is proper of degree e.

8. Let $B \subset \mathbb{C}$ be an annulus bounded by a pair of Jordan curves C_1 and C_2. Prove that there exists an arc $\alpha \subset B$ joining C_1 to C_2, and a loop $\beta \subset B$ separating C_1 from C_2, such that their Euclidean lengths satisfy
$$L(\alpha)L(\beta) \le \operatorname{area}(B).$$
   (Hint: let $f : A(R) \to B$ be a conformal map from a standard annulus $A(R) = \{1 < |z| < R\}$ to B, and consider $\int_{A(R)} |f'(z)| \cdot |z|^{-1} \, |dz|^2$.)