Complex Analysis: Homework 6

1. Show that $2\sum_1^\infty 1/(1+n^2) = \pi\coth(\pi) - 1$.

2. Show that if $\tau \in \mathbb{H}$ is fixed then

$$\theta(z) = \sum_{n=\infty}^{\infty} \exp(2\pi i n^2 \tau)\exp(2\pi i n z)$$

is an entire function of order 2.

3. Prove Wallis's formula

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdots}.$$

4. Find all entire functions $f(z)$ satisfying $f(z+1) = 2f(z)$. Show there are infinitely many such functions of finite order with no zeros and with $f(0) = 1$.

5. For a fixed value of $a > 0$, define for $\operatorname{Re} s > 0$ the function

$$F(s) = \int_0^1 x^s(1-x)^a \frac{dx}{x(1-x)}.$$

Prove $F(s) = \Gamma(a)\Gamma(s)/\Gamma(a+s)$. (Hint: show $G(s) = F(s)\Gamma(a+s)$ satisfies the functional equation $G(s+1) = sG(s)$.)

6. Using the previous result, evaluate $\int_0^1 (1-x^a)^b\,dx$ for $a, b > 0$.

7. Let $\mathcal{O}(\mathbb{C})$ be the ring of all analytic functions on $\mathbb{C}$. Show that the principal ideal (f) is prime iff $(f) = (z-a)$ for some $a \in \mathbb{C}$.

8. Find a conformal homeomorphism from $\mathbb{C} - \overline{\Delta}$ to $\mathbb{C} - E(a,b)$, where $E(a,b)$ is the ellipse $\{x+iy \,:\, x^2/a^2 + y^2/b^2 \le 1\}$.

9. Find a conformal map $f : \mathbb{H} \to \mathbb{H} - E(a,b)$.

10. Find a conformal map $f : \Delta \to (\mathbb{C} - [-1/4, -\infty))$ with $f(0) = 0$ and $f'(0) > 0$, and compute its power series at $z = 0$.