Complex Analysis: Homework 5

1. Prove that for any nonzero polynomial $p(z)$ and any $\lambda \neq 0$, the function $f(z) = p(z) - e^{\lambda z}$ has infinitely many zeros.

2. Let $M(r) = \sup_{|z|=r} |f(z)|$ where $f : \mathbb{C} \to \mathbb{C}$ is an analytic function, not identically equal to zero. Suppose $M(r^2)^2 = M(r)M(r^3)$ for some $r > 0$, $r \neq 1$. Prove that $f(z) = az^n$ for some $a \neq 0$ and integer $n \geq 0$.

3. Prove there is no nonzero analytic function $f : \Delta \to \Delta$ with zeros at the points $a_n = 1 - 1/(n+1)$, $n = 1, 2, 3, 4, \ldots$. (Thus in contrast to $f : \mathbb{C} \to \mathbb{C}$, the zeros of a map $f : \Delta \to \Delta$ cannot be an arbitrary discrete set. Hint: consider $f(0)/B_n(0)$, where $B_n : \Delta \to \Delta$ is a proper map of degree n with zeros at $a_1, \ldots, a_n$.)

4. Formulate and prove an infinite product formula for $\cos(\sqrt{z})$.

5. Give an example of a canonical product $f(z) = \prod_1^\infty (1 - z/a_n)$ that has order exactly 1.

6. Let $f(z)$ be an entire function of finite order with simple zeros at the points $z = n + im$, $(n, m) \in \mathbb{Z}^2$. Show there are polynomials P and Q such that $f(z+1) = e^{P(z)} f(z)$ and $f(z+i) = e^{Q(z)} f(z)$. Prove that at least one of P and Q is nonzero.

7. Prove that $\prod_2^\infty \cos(\pi/2^n) = (2/\pi)$.

8. Show that $1/\Gamma(z)$ has order one, *but* there is no constant $C > 0$ such that $1/\Gamma(z) = O(\exp(C|z|))$.

9. Evaluate $\Gamma(1/3)$. Then find a formula for $\Gamma(3z)$ in terms of Γ at z, $z + 1/3$ and $z + 2/3$.

10. Prove that $\int_0^1 \log \Gamma(t)\,dt = \log \sqrt{2\pi}$, using the duplication formula for $\Gamma(2z)$.