Complex Analysis: Homework 4

1. A holomorphic map $f : \mathbb{H} \to \mathbb{H}$ is a *strict contraction* if there is a $k < 1$ such that $d(f(x), f(y)) < kd(x, y)$ for any $x \neq y$, in the hyperbolic metric. Is it true that every holomorphic map $f : \mathbb{H} \to \mathbb{H}$ is either an automorphism, or a strict contraction?

2. Give a formula for the hyperbolic metric ρ on
$$\Delta^* = \{z \,:\, 0 < |z| < 1\}.$$
(This is the unique metric such that the universal covering map $\pi : (\mathbb{H}, |dz|/y) \to (\Delta^*, \rho)$ is a local isometry.) Prove that a neighborhood of $z = 0$ in Δ^* has finite volume. What does (Δ^*, ρ) look like, geometrically, near $z = 0$?

3. Find a formula for the universal covering map $\pi : \mathbb{H} \to U$, where $U = \{z \,:\, 1 < |z| < R\}$. Then find a formula for the hyperbolic metric on U.

4. Find a metric $\rho(z)|dz|$ on $[-2, 2]$ such that $\pi^*(\rho) = |dz|$ on S^1, where $\pi(z) = z + 1/z$. Then show the Chebyshev polynomials are orthogonal for ρ: that is, $\int_{-2}^{2} P_n(x)P_m(x)\rho(x)\,dx = 0$ if $n \neq m$.

5. Let $f : \mathbb{H} \to \mathbb{H}$ be a nonconstant analytic map of the form
$$f(z) = a_0 z + b_0 + \sum_{1}^{d} -a_i/(z - b_i),$$
where $a_i \geq 0$, $b_i \in \mathbb{R}$ and $(b_1, \ldots, b_d)$ are distinct.
(i) Prove that f is proper. What is its degree?
(ii) Prove that any proper map $f : \mathbb{H} \to \mathbb{H}$ can be expressed in the form above.

6. Let A and K be the rings of analytic and meromorphic functions on $\mathbb{C}$ (under multiplication and addition of functions). Show that K is a field, A is an integral domain, K is the field of fractions of A, and A is integrally closed in K. (The means any $f \in K$ satisfying a monic polynomial $p(X) \in A[X]$ is actually in A.)

   Is K algebraically closed?

7. Let $A(\rho) \subset A$ be the set of entire functions of order $\leq \rho$. Show that $A(\rho)$ is a subring of A, and that $A(\rho)$ is integrally closed in K.

8. Show that if $f \in A(\rho)$ then $f' \in A(\rho)$.

9. What is the order of the entire function $f(z) = \sum z^n/(n^n)^\alpha$, where $\alpha > 0$?

10. Let α be the critical exponent of a sequence $a_n \to \infty$ in $\mathbb{C}$, and let $n(r)$ denote the number of terms with $|a_n| < r$. How is α related to $\alpha' = \limsup_{r\to\infty} \log n(r)/\log r$?