Complex Analysis: Homework 3

1. Determine the Galois group of the field of rational functions $\mathbb{C}(z)$ over $\mathbb{C}$.

2. Suppose $r > 0$ and $(r, ir, -r, -ir)$ are the vertices of a square on $\widehat{\mathbb{C}} \cong S^2$, with internal angles of $2\pi/3$ in spherical geometry. What is r?

3. Which curves on $\widehat{\mathbb{C}}$ are geodesics with respect to the spherical metric $\rho = 2\,|dz|/(1 + |z|^2)$?

4. Prove that the automorphisms of $\widehat{\mathbb{C}}$ preserving the spherical metric coincide naturally with $\mathrm{SU}(2)/\{\pm I\}$, where $\mathrm{SU}(2)$ is the group of linear maps of $\mathbb{C}^2$ preserving $|z|^2 = |z_1|^2 + |z_2|^2$.

5. Pick a basis for the Lie algebra of $\mathrm{SL}_2(\mathbb{C})$, and show how each basis element can be canonically interpreted as a holomorphic vector field on $\widehat{\mathbb{C}} = \mathbb{PC}^2$. Check that Lie bracket corresponds to bracket of vector fields.

6. Determine the automorphism group of the Riemann surface $X = \widehat{\mathbb{C}} - \{0, 1, \infty\}$.

7. Prove there is no proper analytic map $f : \Delta \to \mathbb{C}$. Is there a proper analytic map $\mathbb{C} \to \mathbb{C}^*$ or $\mathbb{C}^* \to \mathbb{C}$?

8. Let $f_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be an unbounded sequence in $\mathrm{Aut}(\widehat{\mathbb{C}})$. Prove that after passing to a subsequence, there are $p, q \in \widehat{\mathbb{C}}$ such that $f_n(z) \to p$ uniformly on compact subsets of $\widehat{\mathbb{C}} - \{q\}$. Give an example of a sequence f_n where $p = q = 0$.

9. Find a metric ρ on $\mathbb{C}$ such that the map $\cos : (\mathbb{C}, |z|) \to (\mathbb{C}, \rho)$ is a local isometry (away from its critical points). Draw a picture of $\mathbb{C}$ in the metric ρ.

10. (a) Prove the Gauss-Bonnet theorem for a hyperbolic triangle $T \subset \mathbb{H}$: $\mathrm{area}(T(a, b, c)) = \pi - a - b - c$, where a, b, c are the interior angles of T.

    (b) Give a formula for the area of a polygon in $\mathbb{H}$ in terms of its external angles.

    Hint for (a): (i) $T(0, 0, 0)$ is an 'ideal triangle' with all vertices at infinity. Show all ideal triangles have area π. (ii) Show *geometrically*

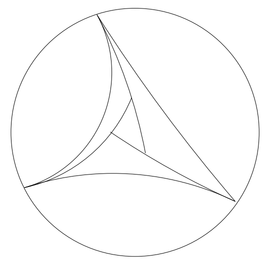

Figure 1.

that $A(a) = T(\pi - a, 0, 0)$ satisfies $A(a + a') = A(a) + A(a')$, and conclude $A(a) = \pi a$. (iii) Extend the sides of $T(a, b, c)$ to rays (Figure 1), and use the corresponding vertices at infinity to relate the area of $T(a, b, c)$ to the area of $T(a, 0, 0)$, $T(b, 0, 0)$ and $T(c, 0, 0)$.

11. (a) State and prove a theorem relating $\mathrm{Aut}(\Delta)$, the subgroup $SU(1,1) \subset \mathrm{SL}_2(\mathbb{C})$ preserving the form

$$\langle Z, W \rangle = Z_1 \overline{W}_1 - Z_2 \overline{W}_2,$$

and the matrices of the form $\left(\begin{smallmatrix} a & b \\ \overline{b} & \overline{a} \end{smallmatrix}\right)$.

(b) Give a formula for the hyperbolic distance $d(z, w)$ on Δ in terms of $\langle Z, W \rangle$, for suitable representatives $Z, W \in \mathbb{C}^2$ of the points $z, w \in \mathbb{PC}^2$.