Complex Analysis: Homework 2

1. Let $p(n)$ be the number of partitions of n into *unequal* parts. (E.g. $p(7) = 5$ because 7 can be written as 7, $1+2+4$, $1+6$, $2+5$ and $3+4$.)

   (i) Show that $f(z) = \prod_1^\infty (1+q^n)$ defines an analytic function on Δ.
   (ii) Show that $f(z) = \sum_1^\infty p(n) q^n$.
   (iii) Show that $f(z)$ cannot be analytically continued beyond the unit disk. (This means if $g(z)$ is analytic on a connected domain $U \supset \Delta$, and $f(z) = g(z)$ on Δ then $U = \Delta$.)

2. The distributional derivative $(d/d\overline{z})(1/z)$ is a constant multiple of the δ-function at the origin; that is, for any compactly supported smooth function ϕ, we have

   $$-\int_{\mathbb{C}} \frac{1}{z}\frac{d\phi}{d\overline{z}}\,dx\,dy = C\phi(0).$$

   Prove this and evaluate C.

3. Let $p(z)$ be a polynomial of degree $d \geq 2$, with distinct roots $r_1, \ldots, r_d$. Show that $\sum 1/p'(r_i) = 0$.

4. Let $E \subset (-1,1)$ be a compact set of zero linear measure, and let $f : \Delta - E \to \mathbb{C}$ be a bounded analytic function. Show that f extends to an analytic function on the whole disk.

5. Let $a_n \in \mathbb{Z}$ be the integer closest to the nth power of the golden ratio $\gamma = (1+\sqrt{5})/2$. Show that $\sum_0^\infty a_n z^n$ is the power series for a rational function $f(z) = P(z)/Q(z)$, and find P and Q.

6. Let $f_n(z)$ be a sequence of polynomials such that $f_n(z) \to f(z)$ pointwise in $\mathbb{C}$. Show there exists a dense, open set $U \subset \mathbb{C}$ such that $f|U$ is analytic. (You may quote the Baire Category theorem.)

7. Give an example of an unbounded analytic function on the unit disk with the property that $\lim_{z\to w} |f(z)| = 1$ for all $w \in S^1$ except $w = 1$.

8. Prove that the space of positive harmonic functions $u(z)$ on Δ with $u(0) = 1$ is compact in the topology of uniform convergence on compact sets. (Hint: if $\operatorname{Re} f(z) > 0$ then $|f/(1+f)| < 1$.)

9. Let $f : \Delta \to \mathbb{C}$ be an analytic function with $f(0) = 0$. Show that $\int_\Delta |\operatorname{Re} f|^2 = \int_\Delta |\operatorname{Im} f|^2$.

10. Show there is an entire function $f : \mathbb{C} \to \mathbb{C}$ such that for any $z \in \Delta$,

    $$\lim_{n\to\infty} f(10n + z) = e^z \quad \text{and} \quad \lim_{n\to\infty} f(10n+5+z) = \sin(z).$$