Complex Analysis Final: Homework 13

All work should be your own. Refer only to class notes (your own and those online) and the course texts. Turn in to McMullen's mailbox (outside room 325) by 11 am on Thursday, 9 December 2010.

1. Recall for any lattice we defined $G_n(\Lambda) = \sum' \lambda^{-2n}$. Since there is a unique modular form of weight 8, there is a universal constant A such that $G_2(\Lambda)^2 = AG_4(\Lambda)$ for all Λ. What is the value of A?

2. What are the orbits of $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{Q} \cup \{\infty\} \subset \partial\mathbb{H}$? What are the orbits of $\Gamma(2)$?

3. For $k \geq 0$, let M_k denote the space of rational forms $f(z)\,dz^k$ on $\widehat{\mathbb{C}}$ with poles of order at most k at $0, 1, \infty$ and no other poles. (i) Determine $\dim M_k$ and give a basis for this space. (ii) Give a finite set of generators for $\oplus M_k$ as a graded ring. (This is the ring of modular functions for $\Gamma(2)$.)

4. Prove that $\lambda(i/2) = 12\sqrt{2} - 16$.

   (Hint: Letting $X_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$, $\lambda(\tau)$ is the cross-ratio of the critical values (suitably ordered) of any degree two map $f_\tau : X_\tau \to \widehat{\mathbb{C}}$. For the square torus, $\tau = i$, choose f_i so its critical values are the roots of $z^4 + 1 = 0$. Then one can choose $f_{i/2}(z) = (f_i(z) + f_i(z)^{-1})/2$, and find that its critical values are $\{-1, -\sqrt{1/2}, \sqrt{1/2}, 1\}$.)

5. Let L be length in the hyperbolic metric of the closed geodesic γ on $\widehat{\mathbb{C}} - \{0, 1, \infty\}$ that makes a figure 8 around 0 and 1. Show that $L = \log(17 + 12\sqrt{2})$. (Hint: show that γ corresponds to a matrix of trace 6 in $\pi_1(X) \cong \Gamma(2)$.)

6. Let $M(x) > 0$ be a continuous function on $\mathbb{R}$.

   (i) Prove there exists an entire function with $|f(x)| > M(x)$ for all $x \in \mathbb{R}$.

   (ii) Prove there exists an entire function with $0 < |g(x)| < M(x)$ for all $x \in \mathbb{R}$.

   (iii) Prove there does not exist an entire function with $|f(z)| > |z|$ for all $z \in \mathbb{C}$.

7. Prove or disprove: $\Gamma'(3/2) = 0$.

8. Let $f : \Delta \to \mathbb{C}$ be analytic and suppose $\int_\Delta |f'(z)|^2\,|dz| < \infty$. Prove that $F(z) = \lim_{r\to 1} f(rz)$ exists and is finite for almost every $z \in S^1$.

9. Let $A = \{z \in S^1 : \mathrm{Im}(z) \geq 0\}$. Find a Riemann map $f : \mathbb{C} - \overline{\Delta} \to \mathbb{C} - A$.

10. Prove that
$$60 \sum_{\lambda \in \mathbb{Z}[i]}' \lambda^{-4} = \frac{64\pi^2\Gamma[5/4]^4}{\Gamma[3/4]^4}.$$