Complex Analysis: Homework 12

1. Let $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ be a lattice in $\mathbb{C}$, with associated Weierstrass $\wp$-function $\wp(z)$.

   Prove there is a unique odd meromorphic function $\zeta(z)$ on $\mathbb{C}$ such that $\zeta'(z) = -\wp(z)$, and relate $\zeta(z)$ to the canonical product $\sigma(z) = z\prod'(1 - z/\lambda)\exp(z/\lambda + z^2/2\lambda)$.

2. Show that $\zeta(z + \lambda_i) = \zeta(z) + \eta_i$ for suitable $\eta_i \in \mathbb{C}$. Using the residue theorem, show these 'dual periods' satisfy
$$\det\begin{pmatrix} \eta_1 & \eta_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} = 2\pi i.$$

3. The canonical product satisfies $\sigma(z + \lambda_i) = \exp(a_i + b_i z)\sigma(z)$. Express a_i and b_i in terms of λ_i and η_i.

4. Suppose $\sum' \lambda^{-4}$ and $\sum' \lambda^{-6}$ are the same for two lattices $\Lambda_1, \Lambda_2 \subset \mathbb{C}$. Does it follow that $\Lambda_1 = \Lambda_2$?

5. Suppose $(x, y) = (\wp(z), \wp'(z))$ satisfies $y^2 = 4x^3 + ax + b$ with $a, b \in \mathbb{R}$, and the polynomial $4x^3 + ax + b = 0$ has only one real root. What can you say about the shape of the lattice Λ used to define $\wp(z)$?

6. Prove that $\wp'(z) = -\sigma(2z)/\sigma(z)^4$.

7. Let $T \subset \mathbb{C}$ be the region bounded by the triangle with vertices $(0, 1, 1 + i)$. Give an explicit formula for a conformal mapping $f : T \to \mathbb{H}$ in terms of the Weierstass $\wp$-function for a suitable lattice.