Complex Analysis: Homework 11

1. Prove there exists a pair of nonconstant meromorphic functions on $\mathbb{C}$ such that $f(z)^3 + g(z)^3 = 1$.

2. Let $\Lambda \subset \mathbb{R}^n$ be a lattice, i.e. a discrete subgroup isomorphic to $\mathbb{Z}^n$. Choose a sequence of vectors $a_1, a_2, \ldots, a_n \in \Lambda$ such that a_1 is a shortest nonzero vector, and (for $i > 1$) a_i is a shortest vector linearly independent from $(a_1, a_2, \ldots, a_{i-1})$.

   Is it always the case then that $\Lambda = \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_n$?

3. Let $\Lambda \subset \mathbb{C}$ be a lattice, let $X = \mathbb{C}/\Lambda$ and let $\mathrm{End}(\Lambda) = \{\alpha \in \mathbb{C} : \alpha\Lambda \subset \Lambda\}$. Show that for each $\alpha \in \mathrm{End}(\Lambda)$, the formula $[f(z)] = [\alpha z]$ defines an analytic covering map $f : X \to X$ of degree $|\alpha|^2$.

   For what values of $\alpha \in \mathbb{C}$ does there exist a lattice with $\alpha \in \mathrm{End}(\Lambda)$?

   Conclude that $\mathrm{End}(\mathbb{Z} \oplus \mathbb{Z}\tau) = \mathbb{Z}$ for almost all values of τ.

4. Where are the zeros of the $\wp$-function for the lattice $\mathbb{Z} \oplus \mathbb{Z}i$? For the lattice $\mathbb{Z} \oplus \mathbb{Z}\omega$? (Here $\omega = (1 + \sqrt{-3})/2$.)

5. Prove that $g_3(i) = 0$ and $g_2(\omega) = 0$. (Recall $g_k(\tau)$ is proportional to $\sum' \lambda^{-2k}$, with $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$.)

6. State and prove a 'double angle' formula for the Weierstrass $\wp$-function. That is, find a rational function $f(z)$ (that may depend on (g_2, g_3)) such that $\wp(2z) = f(\wp(z))$.

7. Let $X = \mathbb{C}/\Lambda$ be a complex torus, and define a map $F : X \to X$ by $[F(z)] = [2z]$. Show that F has a dense orbit on X, i.e. that there exists a $p \in X$ such that $\overline{\{F^n(p) : n > 0\}} = X$, where $F^n(p) = F(F(\cdots F(p)))$.

   Then prove the rational function $f(z)$ of the double-angle formula has a dense orbit on $\widehat{\mathbb{C}}$.

8. For any $\lambda \neq 0, 1, \infty$ let $S_\lambda = \widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$. Let $\mathrm{Aut}(S_\lambda)$ be the group of holomorphic bijections $f : S_\lambda \to S_\lambda$.

   (a) Prove that every $f \in \mathrm{Aut}(S_\lambda)$ is a Möbius transformation.

   (b) Prove that for all λ, $\mathrm{Aut}(S_\lambda)$ contains a subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. How does this group permute the points $\{0, 1, \infty, \lambda\}$?

   (c) Find all values of λ such that $|\mathrm{Aut}(S_\lambda)| > 4$ and identify the group in each case.