Complex Analysis: Homework 10

1. Find a formula for a covering map $f : \mathbb{H} \to \mathbb{C} - S$ whose image is the complement of a square S.

2. Prove that the Little Picard Theorem implies directly that there is a constant $M > 0$ such that any holomorphic map $f : \Delta \to \mathbb{C} - \{0, 1\}$ satisfies $\|f'\| \le M$, where the derivative is measured using the hyperbolic metric on the domain and the spherical metric on the range. (Hint: supposing the derivative tends to infinity, construct a nonconstant entire function $f : \mathbb{C} \to \mathbb{C} - \{0, 1\}$.)

3. Let $T \subset \mathbb{R}^2$ be a (closed) Euclidean triangle, and let $G \subset \mathrm{Isom}(\mathbb{R}^2)$ be the group generated by reflections in the sides of T. (i) Show that the tiles $T_g = \{g(T) : g \in G\}$ cover $\mathbb{R}^2$. (ii) Give an example where every point in $\mathbb{R}^2$ belongs to the interior of at most one tile. (iii) Give an example where every point in $\mathbb{R}^2$ belongs to the interior of infinitely many tiles. (Hint: the closure of G is a Lie subgroup of $\mathrm{Isom}(\mathbb{R}^2)$.)

4. Let f and g be entire functions solving Fermat's equation, $f^n + g^n = 1$, with $n > 2$. Prove that f and g are constant. (Hint: consider f/g).

5. Let $f(z)$ be an entire function such that $f(z)$ is never zero and $f^{-1}(1)$ is finite. Prove that f is constant.

6. Prove there is a unique quadratic differential $q = q(z)\, dz^2$ with double poles at $0, 1, \infty$ with residues A, B, C respectively, and no other singularities. (Recall if $q = (a/z^2 + O(1/z))\, dz^2$ then $\mathrm{Res}_0(q) = a$.)

7. A *circular triangle* is a region $T \subset \widehat{\mathbb{C}}$ bounded by three circular arcs or lines. Let $T_1, T_2 \subset \widehat{\mathbb{C}}$ be two circular triangles with the same interior angles in the same cyclic order. Prove there exists a Möbius transformation sending T_1 to T_2.