

Complex Analysis: Homework 1

Due Tuesday, 14 September 2010

Write your answers neatly on separate sheets, stapled together, with your name on the first page. Clearly indicate (e.g. with a box) the final answer to each problem. All collaborators and references should be attributed.

1. Let $f_n \rightarrow f$ uniformly on compact subsets of an open connected set $\Omega \subset \mathbb{C}$, where f_n is analytic, and f is not identically equal to zero.

(a) Show if $f(w) = 0$ then we can write $w = \lim z_n$, where $f_n(z_n) = 0$ for all n sufficiently large.

(b) Does this result hold if we only assume Ω is open?

2. Let $f(z) = (az+b)/(cz+d)$ be a Möbius transformation. Show the number of rational maps $g: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that

$$g(g(g(g(g(z)))))) = f(z)$$

is 1, 5 or ∞ . Explain how to determine which alternative holds for a given f .

3. Let $\sum a_n z^n$ be the Taylor series for $\tan(z)$ at $z = 0$.

(a) What is the radius of convergence of this power series?

(b) Give an explicit value of N such that $\tan(1)$ and $\sum_0^N a_n$ agree to 1000 decimal places. Justify your answer.

4. Evaluate:

$$\int_{-\infty}^{\infty} \frac{x^6}{(1+x^4)^2} dx.$$

5. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be analytic and let $U \subset \mathbb{C}$ be a bounded region. Suppose $|f(z)|$ is constant on ∂U . Show that either f is constant, or f has a zero in U .

6. Compute the Laurent series centered at $z = 0$ such that

$$\sum_{-\infty}^{\infty} a_n z^n = \frac{1}{z(z-1)(z-2)}$$

in the region $1 < |z| < 2$.

7. Show for any polynomial $p(z)$ there is a z with $|z| = 1$ such that $|p(z) - 1/z| \geq 1$.

8. When is a polynomial $p(x, y)$ expressible in the form

$$p(x, y) = f(x + iy) + g(x - iy),$$

where f and g are holomorphic?

9. Find a smooth function $f(z)$ such that for any compact smooth region $U \subset \mathbb{C}$, we have

$$\text{area}(U) = \int_{\partial U} f(z) dz.$$

10. Give an explicit holomorphic covering map $f : \Delta \rightarrow \Delta^*$, showing that the unit disk is the universal cover of the punctured disk $\Delta^* = \{z : 0 < |z| < 1\}$.