Homework 8
Real Analysis
Math 212a – Harvard University – Fall 1998
Due Friday, 13 November 1998

Royden, Chapter 7: 7, 27, 28, 38.

1. State and prove a version of the Baire category theorem that holds for any compact Hausdorff topological space.

2. Show that the set of irrationals $J = \mathbb{R} - \mathbb{Q}$, with the induced topology, is homeomorphic to a complete metric space.

3. Give an example of a metric space (X, d) which is not homeomorphic to a complete metric space.

   In the next 3 problems, all functions and sequences are complex-valued. For any integrable $f : [0,1] \to \mathbb{C}$, let $T(f) = \langle T_n(f) \rangle$ be the sequence defined by
$$T_n(f) = \int_0^1 f(x) \exp(2\pi i n x)\, dx, \quad n \in \mathbb{Z}.$$

4. Prove that $f \in L^2[0,1] \implies T(f) \in \ell^2(\mathbb{Z})$ and that $T : L^2[0,1] \to \ell^2(\mathbb{Z})$ is an isometric isomorphism of Banach spaces.

5. Show that for $f \in L^1[0,1]$, $T_n(f) \to 0$ as $|n| \to \infty$.

6. Fix any $p > 0$. Show that for a *generic* $f \in L^1[0,1]$ (in the sense of Baire category), we have $\sum |T_n(f)|^p = \infty$.