Math 212a: Advanced Real Analysis

Suggested solutions to Homework 9

Problem 60. Let f_α and f_β denote the indicator functions of $[0, \alpha]$ and $[0, \beta]$ respectively, with $\alpha, \beta > 0$. Show that the ideal $J \subset A = (L^1(\mathbb{R}), *)$ generated by these two functions is dense iff α/β is irrational.

Solution. The Fourier transform of f_α (for simplicity, we forgo the normalization) is given by

$$\int_0^\alpha e^{-ixt} dx = \frac{e^{-ixt}}{-it}\Big|_0^\alpha = \frac{1 - e^{-i\alpha t}}{it}.$$

Note also that $\widehat{f_\alpha}(0) = \alpha \neq 0$. Solving $\widehat{f_\alpha}(t) = 0$ gives $t = 2k\pi/\alpha$ for some integer k. It is then clear that $\widehat{f_\alpha}$ and $\widehat{f_\beta}$ has a common zero if and only if α/β is rational. The result then follows by Weiner's theorem. $\square$

Problem 62. Let $\phi(x) = \cos(x^2) \in L^\infty(\mathbb{R})$. Show that $\lim_{x\to\infty}(\phi * f) = 0$ for all $f \in L^1(\mathbb{R})$, even though $\phi(x)$ does not tend to zero.

Solution. First assume $f = \chi_{[a,b]}$. When $|y|$ is large enough, we have

$$(\phi * f)(y) = \int_a^b \cos(y-x)^2 = \frac{\sin(y-b)^2}{2(b-y)} - \frac{\sin(y-a)^2}{2(a-y)} + \frac{1}{2}\int_a^b \frac{\sin(y-x)^2}{(x-y)^2} dx.$$

It is then clear that the expression above tends to 0 as $y \to \infty$. It then follows that the statement is also true for f simple (i.e., a finite linear combination of indicator functions of intervals). Simple functions are dense in L^1, so we can choose g simple so that $\|f - g\|_1 < \epsilon$. Then $\phi * f = \phi * (f - g) + \phi * g$. Since $\|\phi * (f - g)\|_\infty \leq \|f - g\|_1 < \epsilon$, it then follows that $(\phi * f)(y) < 2\epsilon$ for $|y|$ large enough. The statement then follows. $\square$

Problem 63. Compute $\int_{\mathbb{R}} f(x) - f(x+1) dx$ when (i) $f(x) = 1/x$ for $x > 1$ and 0 elsewhere; (ii) $f(x) = e^{-x}[e^x]$, where $[y]$ is the largest integer not exceeding y.

Solution. (i) We have

$$\int_{\mathbb{R}} f(x) - f(x+1)dx = -\int_0^1 \frac{1}{1+x} + \int_1^\infty \frac{1}{x} - \frac{1}{x+1} dx = \lim_{x\to\infty}(\ln(x) - \ln(x+1)) = 0.$$

(ii) When $x < 0$, $e^x \in (0,1)$ and thus $[e^x] = 0$. Now

$$\int_{\mathbb{R}} f(x) - f(x+1)dx = \lim_{y\to\infty} \int_{-1}^{y} f(x) - f(x+1)dx = -\lim_{y\to\infty} \int_{y}^{y+1} f(x)dx.$$

Since

$$\int_y^{y+1} e^{-x}[e^x] \le \int_y^{y+1} 1dx = 1,$$

and

$$\int_y^{y+1} e^{-x}[e^x] \ge \int_y^{y+1} 1 - e^{-x}dx = 1 - e^{-y} + e^{-y-1}.$$

we conclude that $\int_{\mathbb{R}} f(x) - f(x+1)dx = -1$. □

Problem 75. Prove that if a Banach space A is also an algebra, and multiplication is continuous, then there exists a C such that $\|ab\| \le C\|a\| \cdot \|b\|$ for all $a, b \in A$. (ii) Show that we can find an equivalent norm $|a|$ on A such that $|ab| \le |a| \cdot |b|$. (Equivalent means that $\|a\|$ and $|a|$ induces the same topology on A.)

Solution. Since multiplication is continuous, we have there exists a constant C so that $\|ab\| \le C(\|a\|+\|b\|)$. For any $a \in A$, consider the operator $T_a : A \to A$ defined by $T_a(b) = ab$. Clearly this is a bounded linear operator, and moreover, $\|T_{a/\|a\|}(b)\| \le C(1+\|b\|)$ independent of a. By uniform bounded principle, the operator norms of $T_{a/\|a\|}$ is uniformly bounded above, say by $C' > 0$. Then

$$\|ab\| = \|T_a(b)\| = \|a\| \cdot \|T_{a/\|a\|}(b)\| \le C'\|a\| \cdot \|b\|,$$

as desired. Now define $|a| = C'\|a\|$. Clearly this is an equivalent norm. Moreover $|ab| = C'\|ab\| \le (C')^2\|a\| \cdot \|b\| = |a| \cdot |b|$. □

Problem 76. Let $A = C^k[0,1]$ with $\|f\| = \sup_{i\le k} |f^{(i)}(x)|$. Show by example that A is not a Banach algebra as it stands, and say explicitly how to modify the norm so it becomes one.

Solution. When $k \ge 2$, take $f(x) = x$. Then $\|f\| = 1$. Now $(f^2)(x) = x^2$ and hence $\|f^2\| = 2 > 1 = \|f\|^2$. Hence A is not a Banach algebra. Note that

$$|(fg)^{(i)}(x)| \le \sum_{j=0}^{i} \binom{i}{j} |f^{(j)}(x)||g^{(i-j)}(x)| \le 2^i\|f\|\|g\| \le 2^k\|f\|\|g\|,$$

and so by the previous problem, the norm $\|f\|_1 = 2^k\|f\|$ makes A into a Banach algebra. □

Problem 78. (Functional calculus.) Let A be a Banach algebra (over $\mathbb{C}$), and let $f(t) = \sum a_n t^n$ be an entire function on $\mathbb{C}$. (i) Prove that $F(x) = \sum a_n x^n$ exists for all x and defines an analytic map $F : A \to A$, in the sense that $\phi(F(a+tb))$ is an analytic function of t for all $a, b \in A$ and $\phi \in A^*$. (ii) Prove that $\exp(A) \subset A^\times$ (the invertible elements). (iii) Give examples of algebras of the form $A = C(K)$ to show that sometimes equality holds, and sometimes it does not. (iv) What topological property of K characterizes surjectivity of the exponential map? (Hint: use an appropriate cohomology theory.)

Solution. (i) Let $s_N(x) = \sum_{n=0}^N a_n x^n$. Then for $M > N$, we have

$$\|s_M(x) - s_N(x)\| \le \sum_{n=N+1}^{M} |a_n| \|x\|^n.$$

Over a disk of radius $R > \|x\|$, the Taylor series of f converges absolutely. In particular, we may choose N_0 large enough so that whenever $M > N \ge N_0$, we have $\|s_M(x) - s_N(x)\| < \epsilon$. This shows that $\{s_N(x)\}$ forms a Cauchy sequence, and by completeness, the limit $F(x) := \sum a_n x^n$ exists.

We claim that $\phi(F(a+tb))$ is complex differentiable at $t = 0$ for any $a, b \in A$ and $\phi \in A^*$. Then $\phi(F(a+tb)) = \phi(F(a + t_0 b + (t - t_0)b))$ is complex differentiable at $t = t_0$ for any $t_0 \in \mathbb{C}$, i.e. $\phi(F(a+tb))$ is holomorphic (and hence analytic). We will show as $\epsilon \to 0$,

$$\frac{F(a+\epsilon b) - F(a)}{\epsilon} \to \sum_{n=1}^{\infty} a_n \left(\sum_{k=0}^{n-1} a^k b a^{n-1-k} \right)$$

(Note that the infinite sum exists in A, by the absolute convergence of the series $\sum n a_n x^{n-1}$, the derivative of f.) Assuming this, linearity and continuity of ϕ then gives complex differentiability of $\phi(F(a+tb))$ at the origin. Now for $n \ge 2$,

$$\left\| \frac{(a+\epsilon b)^n - a^n}{\epsilon} - \sum_{k=0}^{n-1} a^k b a^{n-1-k} \right\| \le \sum_{k=2}^{n} \binom{n}{k} |\epsilon|^{k-1} \|a\|^{n-k} \|b\|^k \le 2^{n-1} R^n |\epsilon| \frac{1 - |\epsilon|^{n-1}}{1 - |\epsilon|},$$

where R is chosen so that $R > \max\{\|a\|, \|b\|\}$, and if we choose $|\epsilon| < 1/2$, then the above is bounded above by $(2R)^n |\epsilon|$. Note that $\sum a_n (2R)^n$ converges absolutely. If we set

$$g_{N,\epsilon}(x) := \frac{s_N(a+\epsilon b) - s_N(a)}{\epsilon} - \sum_{n=1}^{N} a_n \left(\sum_{k=0}^{n-1} a^k b a^{n-1-k} \right),$$

then we have $\{g_{N,\epsilon}\}_N$ forms a Cauchy sequence and hence a limit exists. Moreover, taking $N \to \infty$, we have

$$\left\| \frac{F(a+\epsilon b) - F(a)}{\epsilon} - \sum_{n=1}^{\infty} a_n \left(\sum_{k=0}^{n-1} a^k b a^{n-1-k} \right) \right\| \le C|\epsilon|,$$

where $C = \sum |a_n| (2R)^n$. Taking $\epsilon \to 0$, we have the desired result.

(ii) Note that $\exp(x) = \sum x^k / k!$. The inverse is given by $\exp(-x)$. Indeed,

$$\left(\sum_{k=0}^{n} \frac{x^k}{k!} \right) \left(\sum_{l=0}^{n} \frac{(-x)^l}{l!} \right) = \sum_{k,l=0}^{n} \frac{(-1)^l x^{k+l}}{k! l!} = e + \sum_{\substack{1 \le k,l \le n, \\ k+l \ge n+1}} \frac{(-1)^l x^{k+l}}{k! l!}.$$

Here e is the multiplicative identity and the last equality follows from $\sum(-1)^k\binom{n}{k} = 0$. Now note that the tail term has norm bounded by

$$\sum_{s=n+1}^{\infty} \frac{2^s\|x\|^s}{s!} \to 0$$

as $n \to \infty$. This gives the desired identity $\exp(x)\exp(-x) = e$.

(iii) Let $K = [0,1]$. Then $A^\times$ consists of nowhere vanishing continuous functions. Let $f \in A^\times$. Define $g(x)$ as follows: take $g(0)$ to be the value $\log(f(0))$, where log is some branch of the logarithmic function. The logarithmic function can be analytically continued in a neighborhood of $f(0)$ and we can thus define $g(x)$ in an neighborhood of 0. Continue as thus, we have a well defined continuous function g, and $\exp g = f$.

Now set $K = S^1$ the unit circle in $\mathbb{C}$. Consider the function $f(e^{i\theta}) = e^{i\theta}$. This is nowhere vanishing, so it is invertible. On the other hand, it cannot be the exponential of any continuous function. Indeed, if $\exp(g) = f$, where we view g as a periodic function in θ of period 2π. Then $g(\theta) = \theta + 2k_\theta\pi$ for some integer k_θ. By continuity of g, we must have $k_\theta = k$ a constant. But then $g(0) = 0 + 2k\pi \neq 2\pi + 2k\pi = g(\pi)$.

(iv) Consider the short exact sequence of sheaves

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_K \xrightarrow{\exp} \mathcal{O}_K^* \to 0,$$

where $\mathcal{O}_K$ is the sheaf of continuous functions on K, $\mathcal{O}_K^*$ the sheaf of nonvanishing continuous functions, and $\mathbb{Z}$ the locally constant sheaf with integer values. Taking the cohomology long exact sequence, we have

$$0 \to H^0(K,\mathbb{Z}) \to H^0(K,\mathcal{O}_K) \xrightarrow{\exp} H^0(K,\mathcal{O}_K^*) \to H^1(K,\mathbb{Z}) \to 0,$$

where we use the fact that $H^1(K,\mathcal{O}_K) = 0$. Hence exp is surjective if and only if $H^1(K,\mathbb{Z}) = 0$. $\square$