# Math 212a: Advanced Real Analysis

## Suggested solutions to Homework 6

**Problem 35**. Let $\Lambda \subset \mathbb{R}^n$ be a lattice (a discrete subgroup whose quotient $\mathbb{R}^n/\Lambda$ is compact), and let $V$ be the volume of $\mathbb{R}^n/\Lambda$. Let

$$\Lambda' = \{x \in \mathbb{R}^n : \langle x, y\rangle \in \mathbb{Z} \text{ for all } y \in \Lambda\}$$

be the dual lattice, let $f \in \mathcal{S}_n$ be a Schwartz function and normalize the Fourier transform (at variance with Rudin) by:

$$\widehat{f}(t) = \int f(x) \exp(-2\pi \langle x, y\rangle) dx.$$

Prove that:

$$\sum_{\Lambda} f(x) = V^{-1} \sum_{\Lambda'} \widehat{f}(t).$$

*Solution.* Write $\Lambda = \bigoplus_{k=1}^{n} \mathbb{Z}v_k$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Lambda$ be the projection map. Define

$$(\pi_* f)(x) := \sum_{y \in \Lambda} f(x + y).$$

Since $f$ is Schwartz, the series above is absolutely convergent to a continuous function periodic with respect to $\Lambda$ (hence one on $\mathbb{R}^n/\Lambda$). We can also define for a continuous function $g$ on $\mathbb{R}^n/\Lambda$, the pullback $\pi^* g = g \circ \pi$. It is clear that for any $f \in \mathcal{S}_n$ and $g \in C(\mathbb{R}^n/\Lambda)$,

$$(f, \pi^* g) = (\pi_* f, g)$$

where the pairing are defined as the $L^2$ pairing on the corresponding spaces. Set $F = \pi_* f$, then it has a Fourier series expansion (which converges to $x$ pointwise)

$$F(x) = \sum_{t \in \Lambda'} a_t e^{2\pi i \langle t, x\rangle},$$

where

$$a_t = V^{-1} \int_{\mathbb{R}^n/\Lambda} F(x) e^{-2\pi i \langle t, x\rangle} dx = V^{-1}(F, e^{-2\pi i \langle t, x\rangle}) = V^{-1}(f, e^{-2\pi i \langle t, x\rangle}) = V^{-1} \widehat{f}(t).$$

Hence $\sum_{x \in \Lambda} f(x) = F(0) = \sum_{t \in \Lambda'} a_t = V^{-1} \sum_{t \in \Lambda'} \widehat{f}(t)$, as desired. □

**Problem 38**. Show there is a smooth function $f(x)$ on $\mathbb{R}$ that belongs to every Sobolev space $H^s$ but which is not a Schwartz function.

*Solution.* Take $f(x) = 1/(1+x^2)$. This is smooth. Clearly $f(x) = O(1/x^2)$ when $|x| >> 0$. Thus $f \in L^2(\mathbb{R}) = H^0$. It is easily verified that for any integer $n > 0$, we also have $|D^n f(x)| = O(1/x^2)$ when $|x| >> 0$. This implies $D^n f \in L^2(\mathbb{R})$ for all $n > 0$. Thus $f \in H^n$ for all integer $n \geq 0$. From the relation $H^s \subset H^t$ for $s > t$ we immediately conclude that $f$ lies in every Sobolev space. But $f$ is not rapidly decreasing, and hence not Schwartz.

Alternatively, we've calculated $\widehat{f}$ in the last homework. Note that $\widehat{f}(t) = \sqrt{\pi/2}e^{-|t|}$ is rapidly decreasing but not smooth. Thus $f$ is in $H^n$ for all integer $n > 0$ but not Schwartz. □

**Problem 40**. Show that $f(x) = e^x \cos(e^x)$ is a tempered distribution on $\mathbb{R}$, but $g(x) = e^x$ is not.

*Solution.* Let $h(x) = \sin(e^x)$, then $h' = f$. Let $\phi \in \mathcal{S}_1$. We then have, as $h$ is bounded and $\phi'$ is rapidly decreasing,

$$\int f(x)\phi(x)dx = -\int h(x)\phi'(x)dx.$$

Again, as $h$ is bounded, we conclude that the right hand side is bounded in absolute value by $C \sup_{x\in\mathbb{R}} |(1+x^2)\phi'(x)|dx$. This shows that $f$ defines a tempered distribution.

On the other hand, define a function $\phi \in \mathcal{S}_1$ as follows. Choose a positive smooth function $\psi$ compactly supported on $[0, 1/2]$ with $\int \psi = 1$ and define $\phi(x) = \sum_{n=0}^{\infty} \psi(x-n)e^{-x}$. Then

$$\int g(x)\phi(x)dx = \sum_{n=0}^{\infty} 1 = \infty,$$

and hence $g$ is not tempered. □

**Problem 41**. Let $f(x) = C\exp(-x^2/2)$ on $\mathbb{R}$, where $C$ is chosen so that $\|f\|_2 = 1$. (a) Show that the uncertainty principle is sharp for $f$, in the sense that $1/2 = (\Delta P)(\Delta Q)$. (b) Show that if $f \in \mathcal{S}(\mathbb{R})$, $\|f\|_2 = 1$, and $\langle P\rangle = \langle Q\rangle = 0$, and $1/2 = (\Delta P)(\Delta Q)$, then $f = C\exp(-ax^2/2)$ for some $C$ and $a$. (Hint: when is the Cauchy-Schwartz inequality an equality?)

*Solution.* (a) Indeed, as $(Pf)(x) = -if'(x) = iCx\exp(-x^2/2)$, $(Qf)(x) = Cx\exp(-x^2/2)$, we have $Pf = iQf$, $\langle Pf, f\rangle = \langle Qf, f\rangle = 0$, $\langle Qf, Pf\rangle = -\langle Pf, Qf\rangle$. Following the proof of the Uncertainty Principle, we have $1 = 2|\langle Qf, Pf\rangle| = 2\|Qf\|\|Pf\| = 2(\Delta P)(\Delta Q)$.

(b) Going through the proof of the Uncertainty Principle, we need $\langle Pf, Qf\rangle = -\langle Qf, Pf\rangle$, and that $Qf$ is proportional to $Pf$ (since $\|f\|_2 = 1$ and $f \in \mathcal{S}(\mathbb{R})$, neither $Qf$ nor $Pf$ can be the zero function). This gives an ODE $f'(x) + axf(x) = 0$ for some constant $a$. Solving the ODE gives $f(x) = C\exp(-ax^2/2)$ for some constant $C$. Since $\langle Pf, Qf\rangle = -\langle Qf, Pf\rangle$, we need that $a$ is real; but $f$ is also Schwartz, so $a > 0$. □

**Problem 42**. (The quantum harmonic oscillator) Define operators on $f(x) \in \mathcal{S}(\mathbb{R})$ by

$$Hf = x^2 f - D^2 f, \quad Rf = xf - Df, \quad \text{and} \quad Lf = xf + Df$$

where $D = d/dx$. (There are the Hamiltonian, raising and lower operators. The eigenvalues of $H$ are the energy levels of the system, which are quantized)

(a) Prove that $H = (LR + RL)/2$.

(b) Prove that $I = (LR - RL)/2$.

(c) Show that $f_0(x) = \exp(-x^2/2)$ satisfies $Hf_0 = f_0$ and $Lf_0 = 0$.

(d) Define $f_n(x) = R^n(f_0)$ for $n > 0$, and show that $f_n(x) = P_n(x)f_0(x)$ where $P_n(x)$ is a polynomial of degree $n$.

(e) Show that $Hf_n = (1 + 2n)f_n$.

(f) Show that $\widehat{f_n} = (-i)^n f_n$.

*Solution.* (a, b) Given $f \in \mathcal{S}(\mathbb{R})$, we have $L(Rf) = x^2 f - xDf + f + xDf - D^2 f = x^2 f + f - D^2 f$, and $R(Lf) = x^2 f + xDf - f - xDf - D^2 f$. This then gives $(LR+RL)f = 2Ef$ and $(LR - RL)f = 2f$.

(c) $Hf_0 = x^2 \exp(-x^2/2) - D(-x\exp(-x^2/2)) = \exp(-x^2/2) = f_0$. Alternatively, note that $Lf_0 = 0$ and hence $Hf_0 = f_0 + RLf_0 = f_0$.

(d) Inductively, $f_n = xf_{n-1} - Df_{n-1} = (2xP_{n-1} - P'_{n-1})f_0$. Thus $P_n = 2xP_{n-1} - P'_{n-1}$ with $P_0(x) = 1$. The claim then follows.

(e) We can prove that $P'_n = 2nP_{n-1}$ by induction. Indeed, $P_1(x) = 2x$ and $P'_1 = 2P_0$. Moreover, $P'_{n+1} = 2P_n + 2xP'_n - P''_n = 2P_n + 2n(2xP_{n-1} - P'_{n-1}) = (2n+2)P_n$. Thus $Hf_n = x^2 P_n f_0 - P''_n f_0 - 2P'_n f'_0 - P_n f''_0 = (P_n + 2xP'_n - P''_n)f_0 = (P''_{n+1} - P_n)f_0 = (1+2n)f_n$.

Alternatively, note that $Lf_n = LRf_{n-1} = RLf_{n-1} + 2f_{n-1} = R^2 Lf_{n-2} + 4f_{n-1} = \cdots = R^n Lf_0 + 2nf_{n-1} = 2nf_{n-1}$. Hence $Hf_n = f_n + RLf_n = f_n + 2nRf_{n-1} = (1+2n)f_n$.

(f) Note that $\widehat{f_0} = f_0$, and $\widehat{f_n} = \widehat{Rf_{n-1}} = i\widehat{f}'_{n-1} - it\widehat{f}_{n-1} = -iR\widehat{f}_{n-1} = \cdots = (-i)^n R^n \widehat{f_0} = (-i)^n f_n$. □

**Problem 43**. Let $H^s$ denote the Sobolev space on $\mathbb{R}^n$. (a) Show that a tempered distribution $f$ belongs to $H^N$, $N \geq 0$ an integer, if and only if $D^\alpha f \in H^0$ for all $\alpha$ with $|\alpha| \leq N$. (b) Show that $H^0 = (I - \Delta)H^2$. (c) Show that there is no linear differential operator such that $H^0 = P(D)H^1$.

*Solution.* (a) We've shown one direction in class. On the other hand, assume $f \in \mathcal{S}'_n$ such that $D^\alpha f \in H^0 = L^2(\mathbb{R}^n)$ for all $\alpha$ with $|\alpha| \leq N$. Since $\widehat{D^\alpha f} = (it)^\alpha \widehat{f}$, we have

$$\int |t^\alpha|^2 |\widehat{f}(t)|^2 dm(t) < \infty.$$

Here for an multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$. Since

$$(1+|t|^2)^N = (1 + t_1^2 + \cdots + t_n^2)^N = \sum_{|\alpha| \le N} C_\alpha (t^\alpha)^2,$$

we have

$$\int (1+|t|^2)^N |\widehat{f}(t)|^2 dm(t) = \sum_{|\alpha|\le N} C_\alpha \int |t^\alpha|^2 |\widehat{f}(t)|^2 dm(t) < \infty.$$

Hence $f \in H^N$, as desired.

(b) By (a), we have $(I - \Delta)H^2 \subset H^0$. Conversely, take a $L^2$ function $f$. Then $\widehat{f}$ is also $L^2$, and so is $\widehat{f}(t)/(1+|t|^2)$. Let $g \in H^0$ be such that $\widehat{g}(t) = \widehat{f}(t)/(1+|t|^2)$. Clearly $g \in H^2$. Moreover, as distributions $(I - \Delta)g = f$. Thus $H^0 \subset (I-\Delta)H^2$.

(c) Clearly this is possible in dimension $n = 1$. Simply take $P(D) = I + D$. Let us thus assume $n \ge 2$.

Suppose otherwise. By taking Fourier transform, we have $H^0 = P(it)\widehat{H^1}$. First note that the polynomial $P$ has to be linear. Otherwise, suppose $P$ is of degree $k \ge 2$. choose $f \in H^1$ with $\widehat{f}(t) = (1+|t|^2)^{-1/2-n/4-\epsilon}$. Then $f \in H^1$. Consider a change of variable $s_1 = t_1, s_j = t_j - t_1$ for $j \ge 2$. Write $P(it) = \sum_{l=0}^k p_l(s_2, \ldots, s_n) t_1^l$, with $p_k = c \ne 0$ is a nonzero constant, $p_l$ is a polynomial of degree at most $k - l$. Consider the region $\Omega_\delta := \{(t_1, s_2, \ldots, s_n) : t_1 > 0, |s_j| \le \delta t_1\}$. On this region $|P(it)| \ge C t_1^k$ when $t_1$ is sufficiently large and $\delta > 0$ sufficiently small. Now

$$\int_{\Omega_\delta} \frac{|P(it)|^2}{(1+|t|^2)^{1+n/2+\epsilon}} \ge \int_{\Omega_\delta} \frac{C^2 |t_1|^{2k}}{(1+|t|^2)^{1+n/2+\epsilon}} \ge \int_0^\infty \frac{C^2 (2\delta)^{n-1} t_1^{2k+n-1}}{(1+n(1+\delta)t_1^2)^{1+n/2+\epsilon}}$$

which is not integrable when $\epsilon > 0$ is small.

Then $P(it) = P_1(t) + iP_2(t) + c$ where $P_1, P_2$ are homogeneous polynomial of degree 1 with real coefficient; at most one of them is zero. Take $f \in H^0$ with $\widehat{f}(t) = (1+|t|^2)^{-n/4-\epsilon}$. If $P_1$ and $P_2$ are linearly independent, then $P_1(t) + iP_2(t) = -c$ defines a affine subspace. At these points $1/|P(it)|^2$ is not integrable, and hence $\widehat{f}(t)/P(it)$ is not $L^2$, a contradiction.

Otherwise, $|P(it)|^2 \le C(1 + P_1(t)^2)$ for some constant $C$ (WLOG, assume $P_1$ is nonzero). Choose an orthonormal change of coordinate $s = At$ so that $s_1$ is a constant multiple of $P_1(t)$. Define $\Omega = \{s : |s_1| \le |\widetilde{s}|^{1/2}\}$, where $\widetilde{s} = (s_2, \ldots, s_n)$. Then

$$\int_\Omega \frac{(1+|t|^2)^{1-n/2-2\epsilon}}{|P(it)|^2} \ge \int_{\mathbb{R}^{n-1}} \int_{-|\widetilde{s}|^{1/2}}^{|\widetilde{s}|^{1/2}} \frac{C s_1^2}{(1+2|\widetilde{s}|^2)^{n/2+2\epsilon}} ds_1 d\widetilde{s} \ge \int_{\mathbb{R}^{n-1}} \frac{C|\widetilde{s}|^{3/2}}{(1+2|\widetilde{s}|^2)^{n/2+2\epsilon}},$$

which is not integrable when $\epsilon > 0$ is small. □