

# MATH 123 SOLUTIONS PROBLEM SET 1

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## 1. FREE GROUP WORDS

How many reduced words of length  $n$  are there in  $F_3 = \langle a, b, c \rangle$ ?

*Solution.* The identity is the only word of length 0. There are 6 words of length 1,  $a, b, c, \bar{a}, \bar{b}, \bar{c}$ . Any word of length  $n + 1$  is a word of length  $n$  with one of 5 letters added, the only letter not allowed is the inverse of the last letter of the  $n$ -letter word, so for  $n > 0$ , there are  $6 \cdot 5^{n-1}$  reduced words.  $\square$

## 2. PRESENTATION OF $S_3$

Prove  $G = \langle x, y : x^2 = y^2 = e, xyx = yxy \rangle$  is isomorphic to  $S_3$ .

*Solution.* Any word can be reduced to be alternating in  $x$  and  $y$  because  $x$  and  $y$  have order 2. Any word of length more than 4 can be shortened by replacing  $xyx$  or  $yxy$  with the other, and then the outermost letters change and cancel from the previous remark. Therefore,  $G = \{e, x, y, xy, yx, xyx\}$ . Let  $\phi : x \mapsto (12)$  and  $y \mapsto (23)$ . We can see that the relations hold, so this defines a unique homomorphism. Since  $(12)$  and  $(23)$  generate  $S_3$ , it is surjective, and also injective since the cardinalities are finite and match, so it is an isomorphism.  $\square$

## 3. FINITE GROUP PRESENTATION

Does every finite group have a finite presentation?

*Solution.* Let  $X_g$  be a symbol corresponding to all non-identity elements  $g \in G$ . Define

$$H = \{X_g : \forall g, h \in G, X_g X_h = X_{gh}\}$$

is a finitely generated group. Any word can be reduced to length 1. Let  $\phi : X_g \mapsto g$  be a map and the relations hold by construction, so it is a homomorphism. It is surjective and the cardinalities match, so  $\phi$  is an isomorphism from  $H$  to  $G$ .  $\square$

## 4. CONICS

Consider the conic in  $\mathbb{R}^2$  defined by  $x^2 + Axy + y^2 + x = 2$ . For which values of  $A$  is this conic an ellipse? For which values of  $A$  is the conic empty?

*Solution.* The discriminant is  $4 - A^2$ , so this is positive for  $|A| < 2$  and we have an ellipse. For  $|A| > 2$ , it is negative so we have a hyperbola. For  $|A| = \pm 2$ , it degenerates to a parabola.

To verify that it is never empty, observe that when  $(0, \pm\sqrt{2})$  are two solutions, so it is never be empty or a point. It is not a line because  $(0, 0)$  is not in the conic. The only

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*Date:* February 7, 2023.

degenerate case could be when  $A = \pm 2$  and the polynomial decomposes as  $(x \pm y)^2 + x = 2$ . Let  $t = (x \pm y)$  to express this as  $t^2 + x = 2$  showing it is a parabola and not a pair of lines.  $\square$

## 5. QUADRICS

What type of real quadric is the surface defined by  $z^2 + xy = \pm 1$  and by  $x^2 + y^2 + z^2 - xy = 1$ ?

*Solution.* In the first equations, setting  $x = u + v$  and  $y = u - v$  gives  $xy = u^2 - v^2$  putting the equation in the standard form of  $z^2 + u^2 - v^2 = \pm 1$  showing it is a one-sheeted or two-sheeted hyperboloid.

Similarly, for the final case, the same transformation gives the equation expressed as

$$z^2 + (u - v)^2 + (u + v)^2 - (u - v)(u + v) = z^2 + 3v^2 + u^2 = 1$$

which is an ellipsoid.  $\square$

## 6. IRRATIONAL QUADRATIC FORM

Show that the quadratic forms  $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $A_2 = \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}$  are equivalent over  $\mathbb{R}$ , but not over  $\mathbb{Q}$ .

*Solution.* The determinant of  $A_2$  is 23 and its trace is 10. This tells us it has two positive eigenvalues, so the classification of quadratic forms tells us it is equivalent to  $A_1$ . We can compute the eigenvalues as  $5 \pm \sqrt{2}$  (sum to 10, and difference of squares gets  $25 - 2 = 23$ ) which are irrational, so they are not equivalent over  $\mathbb{Q}$ .  $\square$

## 7. BONUS

Prove  $F_2$  and  $F_3$  are not isomorphic.

*Solution.* There are 4 homomorphisms from  $F_2 \rightarrow \mathbb{Z}/2$  and 8 from  $F_3 \rightarrow \mathbb{Z}/2$ .

The above statement is a fully correct proof, but obfuscates two key details. The first is that it is secretly a proof by contradiction. Suppose that there were an isomorphism  $\phi : F_2 \rightarrow F_3$ . Then any homomorphism  $f : F_3 \rightarrow \mathbb{Z}/2$  can be made to be a homomorphism from  $F_2 \rightarrow \mathbb{Z}/2$  by composing with  $\phi$ . A more high-brow way would be to use the Yoneda lemma saying that isomorphic objects (in a locally small category) have isomorphic **hom** sets.

The second hidden statement is actually counting these homomorphisms. Homomorphisms from a free group are classified exactly by where they map the generators, and these are all distinct. In  $\mathbb{Z}/2$ , there are only two elements, so we have two choices for each generator, so in general, there are exactly  $2^n$  homomorphisms from  $F_n \rightarrow \mathbb{Z}/2$ .  $\square$