Homework 9

Algebra II

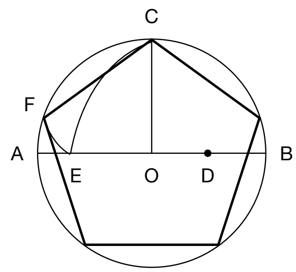

Figure 1. Constructing a regular pentagon. Here D is the midpoint of OB, and the circular arcs CE and EF are centered at D and C respectively.

1. (i) What is a basis for $\mathbb{Q}(2^{1/3})$ over $\mathbb{Q}$?

   (ii) Using division of polynomials, find $q(x) \in \mathbb{Z}[x]$ and $r \in \mathbb{Z}$ such that $(x+1)q(x) = (x^3 - 2) + r$.

   (iii) Express $1/(2^{1/3}+1)$ in the form $a+b\cdot 2^{1/3}+c\cdot 2^{2/3}$, with $a, b, c \in \mathbb{Q}$.

2. Find a subring A of the ring of matrices $\mathrm{M}_2(\mathbb{R})$ such that A is isomorphic to $\mathbb{C}$. Justify your example. (Hint: find a matrix J such that $J^2 = -I$.)

3. Let $p(x) = x^3 + x^2 - 2x - 1$.

   (i) What are the roots of the equation $p(z + 1/z) = 0$?

   (ii) Prove that $t = 2\cos(2\pi/7)$ and $s = 2\cos(4\pi/7)$ are roots of $p(x)$.

   (iii) Prove that $\mathbb{Q}(t)$ contains s.

   (iv) Prove that $\mathbb{Q}(t)$ is the splitting field for $p(x)$.

4. Prove that the construction shown in Figure 1 really does produce a regular pentagon.

5. Let $L = \mathbb{C}(t)$ be the field of rational functions over $\mathbb{C}$, and let $K = \mathbb{C}(t^2) \subset L$. Prove that L is isomorphic to K. What is $[L : K]$?

6. (Bonus problem.) Prove that if $p = 2^n + 1$ is prime, and $n > 0$, then n is a power of 2.