Homework 1
Algebra II

Important. All homework must be typeset, or handwritten very clearly, on sheets of paper that are *stapled together*, and handed in at the start of class on Tuesdays. You may collaborate with others, but you must write up your solutions on your own, and attribute your collaborators.

1. Let G be the free group $\langle a, b, c\rangle$. How many reduced words of length n are contained in G?

2. Let $G = \langle x, y \,:\, x^2 = y^2 = e, xyx = yxy\rangle$. Prove that G is isomorphic to the symmetric group S_3. (Hint: to show $|G| \leq 6$, first show that $(xy)^3 = e$.)

3. Does every finite group have a finite presentation?

4. Consider the conic in $\mathbb{R}^2$ defined by $x^2 + Axy + y^2 + x = 2$. For which values of A is this conic an ellipse? For which values of A is the conic empty?

5. What type of real quadric is the surface defined by $z^2 + xy = 1$? By $z^2 + xy = -1$? By $x^2 + y^2 + z^2 - xy = 1$? (Possible answers: ellipsoid, 1–sheeted hyperboloid, 2–sheeted hyperboloid.)

6. Consider the quadratic forms defined by the matrices $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 4 & 1 \\ 1 & 6 \end{smallmatrix}\right)$. Show that A_1 and A_2 are equivalent over $\mathbb{R}$, but not over $\mathbb{Q}$. (Equivalent over k means $B^t A_1 B = A_2$ for some $B \in \mathrm{GL}_2(k)$.)

7. (Bonus problem — optional.) Prove that the free groups $F_2 = \langle a, b\rangle$ and $F_3 = \langle a, b, c\rangle$ are not isomorphic. (A brief, correct answer is acceptable.)