Homework 3
Sets, Groups and Knots
Math 101 – Harvard University

1. Suppose A and B are finite sets, with $|A| = a$ and $|B| = b$.

   (i) Give the definition of a relation between A and B.

   (ii) How many possible relations are there between A and B?

2. (i) Prove that for any function $f : X \to Y$, we have
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$
   for all $A, B \in \mathcal{P}(Y)$.

   (ii) Show by an example that it is *not* true in general that $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \in \mathcal{P}(X)$.

   (iii) What natural hypothesis on f *does* imply that $f(A \cap B) = f(A) \cap f(B)$?

3. Let $i, j \in \mathbb{N}$ be natural numbers. Considering i and j as sets, let
$$A = i - j = \{x \in i \,:\, x \notin j\}.$$
   When is $A \in \mathbb{N}$? When A is a natural number, what number is it?

4. Let $\mathbb{Z} = \mathbb{N} \times \mathbb{N}/\sim$, where $(a, b) \sim (c, d)$ if $a + d = b + c$. Noting that $(a - b)(c - d) = (ac + bd) - (bc + ad)$ we define
$$(a, b) * (c, d) = (ac + bd, ad + bc).$$
   Prove that if $(a', b') \sim (a, b)$, then $(a', b') * (c, d) \sim (a, b) * (c, d)$. (This shows that multiplication is well–defined on $\mathbb{Z}$.)

5. Prove that if A and B are finite sets, then $A \cup B$ is finite. (Note: A and B may overlap.)

6. Prove that if $F \subset \mathbb{N}$ is a finite set, then $|\mathbb{N} - F| = |\mathbb{N}|$.

7. Prove that if $|A| = |B|$ then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

8. (Bonus.) Let A be a set with an equivalence relation $\sim$. Show there is an 'election' function $f : A \to A$ such that for all $x, y \in A$, (i) $f(x) \sim x$, and (ii) if $x \sim y$ then $f(x) = f(y)$. (One can think of $f(x)$ as the 'president' of the country whose citizens are the equivalence class $[x]$.)