Course Outline
Complex manifolds
Math 241, Fall 1992, Berkeley CA
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Texts: Griffiths and Harris, *Principles of algebraic geometry.* Griffiths, *Introduction to algebraic curves.* Farkas and Kra, *Riemann surfaces.* Forster, *Lectures on Riemann surfaces.*

1. Definition of complex manifolds; examples: $\mathbb{P}^n$, $\mathbb{C}/\Lambda$. Theorem: a holomorphic function on a (connected) compact complex manifold is constant.

2. Line bundles. Transition function: every line bundle can be specified by a covering $< U_\alpha >$ of X, and holomorphic maps
$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*,$$
satisfying $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ (the cocycle condition.) A *section* is specified by $\sigma_\alpha : U_\alpha \to \mathbb{C}$ such that $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$.

3. Examples of line bundles: the tautological bundle over $\mathbb{P}^n$.

4. Transition functions for the tautological bundle: given homogeneous coordinates $[z_0 : \ldots : z_n]$ on $\mathbb{P}^n$, and charts $U_i = \{z_i \neq 0\}$ mapped to $\mathbb{C}^n$ by $(z_0/z_i, \ldots, \widehat{z_i/z_i}, \ldots, z_n/z_i)$, we have $g_{ij} = z_i/z_j$.

5. Theorem: the tautological bundle over $\mathbb{P}^n$ admits no nonzero holomorphic section.

6. Constructions with line bundles and vector bundles: direct sum, tensor product, exterior powers. Pullbacks of bundles.

7. The holomorphic tangent and cotangent bundles of a complex manifold. The canonical bundle $K_X = \bigwedge^n T^*X$.

8. Analytic hypersurfaces and divisors. The line bundle determined by a divisor: write $D = (f_\alpha)$ on each element of a covering U_α, and define $\mathcal{L}_D$ by the transition functions $g_{\alpha\beta} = f_\alpha/f_\beta$.

9. Sheaf cohomology. $\mathrm{Div}(X) = H^0(X, \mathcal{M}^*/\mathcal{O}^*)$. The natural maps

$$\mathcal{M}^*(X) \to \mathrm{Div}(X) \to \mathrm{Pic}(X)$$

and their relation to the exact sequence of sheaves

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0.$$

10. The bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$ determined by the divisor dH, H any hyperplane.

11. The tautological bundle on $\mathbb{P}^n$ is isomorphic to $\mathcal{O}(-1)$. The canonical bundle is isomorphic to $\mathcal{O}(-n-1)$.

12. The space $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is isomorphic to the space of polynomials on $\mathbb{C}^{n+1}$ that are homogeneous of degree d.

    Proof. A homogeneous holomorphic map $f : \mathbb{C}^{n+1} - 0 \to \mathbb{C}$ extends across the origin by Hartog's theorem. Developing f in a power series, we find only one degree of monomial can occur.

13. Effective divisors and linear systems. The base locus of a linear system. Correspondence between $H^0(X, \mathcal{O}(D)$ and meromorphic functions such that $(f) + D \geq 0$.

14. Theorem: there is a natural correspondence between (nondegenerate holomorphic maps $X \to \mathbb{P}^n$, up to composition with automorphism of projective space), and (line bundles $\mathcal{L}$ over X together with a subspace E of $H^0(X, \mathcal{O}(\mathcal{L}))$ such that the linear system $|E|$ is basepoint free.

15. Example: the complete linear system $|dP|$, where P is a point on $\mathbb{P}^1$, embeds $\mathbb{P}^1$ in $\mathbb{P}^d$ as a rational normal curve. Theorem: Every irreducible nondegenerate curve of degree d in $\mathbb{P}^d$ is a rational normal curve.

16. Example: the Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$ is associated with the complete linear system $|2H|$, H a hyperplane. Theorem: the chordal variety of the Versonese surface has dimension 4.

17. Further discussion of the chordal variety: every projective complex manifold of dimension n embeds in $\mathbb{P}^{2n+1}$.

18. Degree of projective varieties; resultants. To find if $f(x)$ and $g(x)$ (polynomials) have a common zero, try to solve for $r(x)$ and $s(x)$ of the appropriate degree such that $rf + sg = 0$. This leads to a system of linear equations whose determinant is the resultant of f and g.

19. Bezout's theorem: given curves C and D of degrees c and d in $\mathbb{P}^2$, either the intersection $C \cdot D$ has cd points counted with multiplicity, or C and D have a component in common.

20. Since $\mathrm{GL}_n \mathbb{C}$ is connected, every complex manifold has a canonical orientation, and all intersections count positively. The cohomology ring of projective space.

21. Adjunction formulas. Let X be a hypersurface in a complex manifold Y. I. The normal bundle $N = T_Y/T_X$ is isomorphic to $\mathcal{L}_X|X$. II. $K_X = (K_Y \otimes \mathcal{L}_X)|X$.

22. Examples. If $X \subset \mathbb{P}^2$ is a curve of degree d, then K_X is the restriction of $\mathcal{O}(d-3)$.

23. Theorem of Pappus and Pascal. Let D_1 and D_2 be two cubic curves in the projective plane, such that six points of $D_1 \cap D_2$ lie on a conic C. Then the remaining three points lie on a straight line. Proof: let E be an element of the linear system generated by the D_i that meets C in a seventh point P. By Bezout, E and C must share a component. Hence the remaining component of E is a line. (If C is a union of two lines, take P to be their point of intersection.)

    Corollary: the intersections of the opposite sides of a hexagon inscribed in a conic lie on a straight line.

24. The classification of compact complex Riemann surfaces X. Any such X is projective, admits an embedding in $\mathbb{P}^3$ and an immersion in $\mathbb{P}^2$ with only ordinary double points.

25. The arithmetic genus $g_a(X)$ is the dimension of the space $H^0(X, K_X)$ of holomorphic 1-forms on X. Theorem: If $X \subset \mathbb{P}^2$ is a smooth curve of degree d, then $g_a(X) \geq (d-1)(d-2)/2$. Proof: This is the dimension of the space of section of $\mathcal{O}(d-3)$ on $\mathbb{P}^2$, and since X is not contained in a curve of degree $d-3$, the restriction map is injective.

26. The topological genus $g_t(X)$ is one half the dimension of $H^1(X, \mathbb{C})$. It is related to the Euler characteristic by $\chi(X) = 2 - 2g_t(X)$. Theorem: $g_a(X) \leq g_t(X)$. Proof: A holomorphic 1-form is closed. A closed form is exact iff all its periods are zero. The sum of a holomorphic and an antiholomorphic 1-form is exact iff both are zero.

27. The Riemann-Hurwitz theorem: for any holomorphic map $f : X \to Y$ of degree d between compact Riemann surfaces, $d > 0$, we have

$$\chi(X) = d\chi(Y) - b$$

where b is the number of critical points of f (counted with multiplicity).

28. Theorem: A smooth curve X of degree d in $\mathbb{P}^2$ has $g_t(X) = (d-1)(d-2)/2$. Proof: project to a line and apply Riemann-Hurwitz. Corollary: $g_t = g_a$ for smooth curves in the projective plane.

29. For curves in the plane with ordinary double points, a modification of the above computation shows $g_t = g_a = (d-1)(d-2)/2 - \delta$, where δ is the number of double points. Since every compact Riemann surface can be realized in this way, the topological and arithmetic genus always agree.

30. In the process we have proven special cases of the de Rham theorem ($H^1_{DR}(X, \mathbb{C}) = H^1(X, \mathbb{C})$) and the Hodge theorem ($H^1_{DR}(X) = \Omega(X) \oplus \overline{\Omega(X)}$, where $\Omega(X)$ denotes the space of holomorphic 1-forms).

31. Theorem: the degree of the canonical bundle on a compact Riemann surface is $2g - 2 = -\chi(X)$. Proof: map to $\mathbb{P}^1$ and pull back a section.

32. Hyperelliptic Riemann surfaces. Any set $B \subset \mathbb{P}^1$ of even cardinality determines a *hyperelliptic* Riemann surface X and a map $f : X \to \mathbb{P}^1$ of degree two branched exactly over B. The genus of X is $g = |B|/2 - 1$. Assuming $B \subset \mathbb{C}$, the space $\Omega(X)$ of holomorphic one forms on X is given by

$$\frac{P(z)dz}{\sqrt{Q(Z)}},$$

where Q is a polynomial with simple zeros on B and P ranges over all polynomials of degree at most $g - 1$.

33. Theorem: Every Riemann surface of genus 1 is isomorphic to $\mathbb{C}/\Lambda$ for a lattice Λ. Proof: Let ω be a nonwhere vanishing holomorphic one-form on X. Then $\int_{z_0}^{z} \omega = f(z)$ is a well-defined holomorphic function on the universal cover of X, providing an isomorphism to $\mathbb{C}$.

34. Theorem: Any compact connected complex group G is abelian. Proof: The adjoint map sends G into the group of automorphism of $\mathfrak{g}$, and this holomorphic map must be constant because G is compact.

35. Poncelet's Theorem: Given two conics C and D, and a point p on C, there is an n-gon with a vertex at p, inscribed in C and circumscribed in D, iff the same is true for all points p.

    Proof: The configuration of C and D defines a fixed-point free automorphism of the elliptic curve E covering C and branched over $C \cap D$. The nth iterate of this automorphism has a fixed point iff it is the identity.

36. The Residue Theorem: $\sum_X \operatorname{Res}(\omega, x) = 0$ for any meromorphic 1-form on a compact Riemann surface X. Proof: a holomorphic one form is closed; apply Stokes' theorem.

37. Theorem (Riemann-Roch): For any line bundle L on a Riemann surface X of genus g,

$$\dim H^0(X, L) = \deg L - g + 1 + \dim H^0(X, K_X \otimes L^*).$$

    Idea: the residue theorem provides the only obstruction to the existence of a meromorphic function. This idea leads to a rigorous inequality in one direction for $L = \mathcal{L}_D$ and D an effective divisor. In this inequality, each term has an interpretation: $\deg L$ is the dimension of the space of potential Laurent series along D; g is the dimension of the space of linear conditions determines by applying the residue theorem to $f\omega$ for each holomorphic one-form ω; 1 comes from the constant functions, which have trivial Laurent tails; and $\dim H^0(X, K_X \otimes L^*)$ is the space of one-forms which determine no constraint on the Laurent series.

38. Examples: if X has genus 0, $\dim H^0(X, L_P) = 2$, so X admits a degree one meromorphic function, and therefore X is isomorphic to $\mathbb{P}^1$.

    If X has genus 1, then $\dim H^0(X, L_{2P}) = 2$, so X is hyperelliptic. Similarly for genus two.

39. Theorem: Let X be a curve of genus $g \geq 2$. Then either (a) X is hyperelliptic, and the canonical map factors through $\mathbb{P}^1$ which maps to $\mathbb{P}^{g-1}$ as a rational normal curve, or (b) X embeds in $\mathbb{P}^{g-1}$ under the canonical map as a curve of degree $2g-2$.

40. Riemann's count: a compact Riemann surface X of genus $g > 1$ depends on $3g-3$ parameters.

    Heuristic argument: choose any degree $d > 2g$. By Riemann-Roch, any X of genus g admits a meromorphic function $f : X \to \mathbb{P}^1$ of degree exactly d. By Riemann-Hurwitz, the number b of branch points of f is $b = 2g - 2 + 2d$. Consider the space $\mathcal{F}$ of all pairs (X, f) as above. By assigning to (X, f) the branch locus $B \subset \mathbb{P}^1$ of f, we obtain a map $\mathcal{F} \to (\mathbb{P}^1)^(b)$ that is finite to one.

    This map is surjective by a topological argument (given B, we can construct a degree d cover of the required genus). This $\dim \mathcal{F} = b$.

    Now project $\mathcal{F}$ to $\mathcal{M}_g$, the moduli space of curves of genus g, by forgetting f. The fibers of this map correspond to the choice of (a) an effective divisor D on X degree d, to play the role of $f^{-1}(\infty)$; and (b) a (generic) element of $H^0(X, \mathcal{O}(D))$. Choice (a) depends on d parameters, while Riemann-Roch shows choice (b) depends on $1 - g + d$ parameters. Putting it all together, we find (choice of branch locus) = (choice of curve X)+ (divisor on X) + (section of $\mathcal{O}(D)$), so $b = \dim \mathcal{M}_g + d + 1 - g + d = 2g - 2 + 2d$, and thus $\dim \mathcal{M}_g = 3g - 3$.

41. The Uniformization Theorem: any simply connected Riemann surface is isomorphic to $\mathbb{H}$, $\mathbb{C}$ or $\widehat{\mathbb{C}}$. The proof entails the construction of first harmonic functions, then meromorphic functions.

42. A function u on a Riemann surface X is harmonic if $d * du = 0$. Equivalently, $\overline{\partial}\partial u = 0$; thus $*du$ is a closed 1-form, and ∂u is a holomorphic 1-form.

    Theorem: $u = \operatorname{Re} f$, $f : X \to \mathbb{C}$ holomorphic, if and only if the form $*du$ has no periods.

    Proof: Write $*du = dv$ and set $f = u + iv$.

    Corollary: Any harmonic function on the punctured disk Δ^* can be written

$$u = \mathrm{Re}(\alpha \log z + \sum_{-\infty}^{\infty} a_n z^n).$$

Proof: Use the log term to remove the period of $*du$.

43. The Dirichlet Problem on a region Ω in a Riemann surface X is to extend a continuous function on $\partial\Omega$ to a harmonic function on Ω.

    Theorem: Dirichlet Problem has a solution on the unit disk Δ. Proof: given f on the boundary of the disk, the extension F must satisfy $F(0)$ is equal to the average of f over the boundary. Applying this principle plus conformal invariance leads to the Poisson integral formula

$$F(z) = \int_{S^1} \frac{1 - |z|^2}{|w - z|^2} f(w) \frac{|dw|}{2\pi}.$$

    This can be interpreted, using hyperbolic geometry, as saying $F(z)$ is the visual average of f as seen from z.

44. Harnack's inequality: for any compact subset K of a Riemann surface X, a positive harmonic function u on X satisfies for all $x, y \in K$, $1/C_K \le u(x)/u(y) \le C_K$. This can be proved by locally applying the Schwarz inequality to $u + iv$, a holomorphic map into the right half-plane.

45. Harnack's principle: Let u_n be an increasing sequence of harmonic functions on X. Then either $u_n \to \infty$ uniformly on compact sets, or $\lim u_n = u$ is harmonic.

46. Perron's principle: let $\mathcal{F}$ be a nonempty family of subharmonic functions on a Riemann surface X, such that for all u and v in $\mathcal{F}$, (a) $\max(u, v) \in \mathcal{F}$, and (b) for any conformal disk Δ in X, the function obtained by changing u on Δ to its harmonic extension from $\partial\Delta$ is also in $\mathcal{F}$.

    Then $\overline{u}(P) = \sup_{\mathcal{F}} u(P)$ is either identically infinity, or it is a harmonic function on X.

47. A *barrier* β at a point P in $\partial\Omega \subset X$ is a continuous function defined on $U \cap \overline{\Omega}$ for some neighborhood U of P, such that *beta* is subharmonic on $U \cap \Omega$, $\beta(P) = 0$ and $\beta < 0$ at other points.

A point of $\partial\Omega$ is *Dirichlet regular* if it admits a barrier. If Ω is bounded by smooth curves, then every boundary point in regular.

48. Theorem: The Dirichlet problem is solvable for every bounded f in $C(\partial\Omega)$ if and only if every boundary point of Ω is regular. Moreover the solution can be chosen with $\inf f \le F \le \sup f$.

    Proof: In one direction, extending a continuous function peaked at P shows every point in the boundary is regular.

    For the other direction, consider the Perron family $\mathcal{F}$ of subharmonic functions continuous on $\overline{\Omega}$, lying between the max and min of f on Ω, and lying below f on $\partial\Omega$. Thus sup of this family solves the problem.

    Remark: if $\overline{\Omega}$ is compact, the Dirichlet problem has a unique solution by the maximum principle. However existence follows even without compactness.

49. Classification of Riemann surfaces in terms of potential theory: X is *elliptic* if it is compact; X is *hyperbolic* if it is noncompact and it carries a nonconstant negative subharmonic function; X is *parabolic* otherwise. (This should not be confused with the classification of X in terms of the type of its universal cover.)

    The classical *type problem* is to determine whether a concretely given X is parabolic or hyperbolic. More geometrically, X is parabolic if and only if Brownian motion is recurrent. (For example, $\mathbb{C}$ is parabolic and $\mathbb{H}$ is hyperbolic.)

50. If X is hyperbolic, every compact set K on X with regular boundary admits a *harmonic measure*, that is a nonconstant positive minimal harmonic function equal to 1 on ∂K. (Minimal means the function lies below any other with the same properties).

    The proof is to consider a Perron family, and use the existence of a positive superharmonic function to conclude that the sup is nonconstant. Alternatively, solve the Dirichlet problem with boundary values 0 and 1 on a sequence of regions exhausting $X - K$.

51. A *Green's function* $G_P(z)$ on a hyperbolic Riemann surface X is a minimal positive harmonic function on $X - \{P\}$ such that for a local parameter with $z(P) = 0$, $G_P + \log|z|$ extends to a harmonic function across P. (This means G_P has a sink at P).

52. Theorem: X admits a Green's function at $P \implies X$ is hyperbolic $\implies$ existence of harmonic measures $\implies$ existence of Green's functions at all points P.

    The main point is the last. Work with the Perron family of compactly supported positive subharmonic functions such that $u + \log|z|$ is subharmonic near P. We must find a bound for u on compact subsets of $X - P$. Let γ_r be the sup of u over $|z| = r$. Then $\gamma_r \le \gamma_1 + \log(1/r)$, so it suffices to bound γ_1. To this end, let h be the harmonic measure of $|z| \le 1/2$, and let $\sigma < 1$ be the sup of h over $|z| = 1$. Then one may check that $\gamma_{1/2} h \ge u$, so $\gamma_1 \le \sigma\gamma_{1/2} \le \gamma_1 + \log(2)$; this gives a bound for γ_1.

53. Theorem: a hyperbolic Riemann surface admits a nonconstant meromorphic function. Proof: let u and v be the Green's functions at two distinct points P and Q; then $f = \partial u/\partial v$, the ratio of two meromorphic one-forms, works. (Note that near P, the Green's function looks like $-\log|z|$ + harmonic, so ∂u has a simple pole at P.)

54. Theorem: If X is hyperbolic and $H^1(X) = 0$, then X is isomorphic to the unit disk.

    Proof: For each P in X, let $G_P(z)$ be the Green's function with a pole at P. Then we can combine G_P with its (multivalued) harmonic conjugate to obtain a holomorphic map $f_P : X \to \Delta$ such that $-\log|f_P(z)| = G_P(z)$. (That is, $f(z) = \exp(-G_P - G_P^*)$.) It will turn out that f_P is an isomorphism sending P to 0.

    For Q in X consider $g = M \circ f_P$ where M is an automorphism of the disk exchanging $f_P(P)$ and $f_P(Q)$. Then it is not hard to see $-\log|g|$ lies above G_Q, and hence $|g(z)| \le |f_Q(z)|$. In particular, this implies $|f_P(Q)| = |f_Q(P)|$, since $|g(P)| = |M(f_P(P))| = |f_P(Q)|$.

    Thus we can form the holomorphic function $h = g/f_Q$ mapping X into the disk; since $|h(P)| = 1$, we have g is equal to a constant times f_Q. But g vanishes at $f_P^{-1}(f(Q))$, while f_Q vanishes only at Q. Thus f is injective. Applying the Riemann mapping theorem, we have that X is isomorphic to the disk. (In fact, minimality of the Green's function implies f_P is an isomorphism onto the disk.)

55. The case of a general Riemann surface can now be handled with some input from the theory of univalent functions.

Let X be simply connected and non-compact. Then X can be exhausted by an increasing collection of simply-connected subsurfaces Ω_n with $\overline{\Omega_n}$ a compact submanifold. It is easy to see each Ω_n is a hyperbolic surface, by solving the Dirichlet problem on $\Omega_n - D$ for a disk D, with boundary values 0 and 1. Thus each region Ω_n is isomorphic to the unit disk Δ.

Let p be a basepoint in Ω_1 and let v be a vector in the tangent space to p. Let f_n be the unique isomorphism from Ω_n to the disk $\Delta(R_n)$ of radius R_n, sending p to the origin and v to unit vector in the positive direction.

Since each region Ω_i is isomorphic to the disk, $f_n|\Omega_i$ is a normalized family of univalent functions, so it has a convergent subsequence. The limit is injective on each one so it is injective on X. Thus X is isomorphic to a plane domain, and thus to $\mathbb{C}$ or Δ.

Finally suppose X is compact and simply-connected. Then $X - \{p\}$ is simply-connected, and it is not hard to it is isomorphic to $\mathbb{C}$; then X itself is isomorphic to $\widehat{\mathbb{C}}$.

56. Theorem. A compact Riemann surface is covered by $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}$ if its genus is 0, 1 or ≥ 2 respectively.

    This can be proved by pure topology, since the classification also agrees with that of π_1 as trivial, abelian or nonabelian.

57. Theorem. An automorphism of a compact Riemann surface X is either linear ($g = 0$), a translation ($g = 1$), or of finite order (any genus).

    Proof. For $g > 1$ in fact the automorphism group of X is finite. One proof is to use the fact that the automorphism group is by isometries (in the hyperbolic metric), so it is *compact*; and the identity has negative self-intersection, so it is *discrete*. (Of course with discreteness in hand one can form the orbifold $X/\operatorname{Aut}(X)$, and show its Euler characteristic is at least as negative as the $(2, 3, 7)$-orbifold on the sphere, which has $\chi = -1/42$; thus $|\operatorname{Aut}(X)| \leq 84(g - 1)$ (Hurwitz).

58. Theorem (Belyi). A Riemann surface X is defined over a number field iff it admits a map to $\mathbb{P}^1$ branched over no more than 3 points.

    Proof, in one direction. Suppose X is defined over a number field. Then it is branched over a finite set B of algebraic points. We then find a composition of polynomial maps $P : \mathbb{P}^1 \to \mathbb{P}^1$ such that $P(B)\cup$ (the critical values of P) is $\{0, 1, \infty\}$.

First, if x in B is of degree d over the rationals, then we can find a rational polynomial P of degree d vanishing at x. The critical values of P have degree at most $d-1$ over the rationals, so we can eventually arrange that B lies in $\mathbb{Q}$.

Second, consider the polynomial $P(z) = az^n(1-z)^m$. This has a critical points only at 0, 1, ∞ and $n/(m+n)$, and by choosing a appropriately we can arrange that the critical values are 0, 1 and ∞. Thus P reduces the number of points in B by one, by removing the critical value at $n/(m + n)$. Eventually we reduce to $B = \{0, 1, \infty\}$.

Proof, the other direction. X and its branched cover can be defined over some finitely generated extension of $\mathbb{Q}$. Think of the transcendentals in this extension as variables parameterizing a family of Riemann surfaces. Since the triply punctured sphere is rigid, the isomorphism type of X is constant in any component of this family. So by specializing these values to algebraic numbers, we obtain a curve defined over a number field.

59. Remark: Belyi's theorem shows that X is defined over a number field iff it can be built out of equilateral triangles. One direction is by pulling back a triangulation of the triply-punctured sphere; the other is by making a barycentric subdivision if necessary, then "folding up" to get $\mathbb{P}^1$.

60. Higher dimensions; Kodaira dimension κ, which is the maximal dimension of the image of X under a pluricanonical embedding, or $-\infty$ if no power of the canonical embedding has a section. For Riemann surfaces, the classification by universal cover agrees with $\kappa = -\infty, 0, 1$.

61. Surfaces; blowing points down; minimal surfaces. Products of curves and bundles over curves.

62. Classification of automorphisms. A surface of Kodaira dimension $-\infty$ is rational. For Kodaira dimension 1, the surface admits a canonical map to curve, so its automorphism group reduces to that of a curve (and the fibers). For dimension 2, the automorphisms extend to maps on projective space via a pluricanonical embedding, and the automorphism

group is finite. Thus the remaining cases of interest have dimension zero.

63. Definition. A $K3$-surface is a simply-connected surface with trivial canonical bundle.

    Theorem. Every minimal surface of Kodaira dimension zero is isomorphic to an abelian variety, a $K3$-surface, or a finite quotient of one of these.

64. Examples of K3-surfaces: The intersection of a $(1,1)$ and a $(2,2)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$. A $(2,2,2)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

    Both have lots of automorphisms, coming from involutions by projection to the factor projective spaces. For example, the $(2,2,2)$-case is like a family of $(2,2)$-curves on a quadric (in 3 different ways), and each has a Poncelet-like automorphism group.