Entropy, Algebraic Integers, and moduli of surfaces

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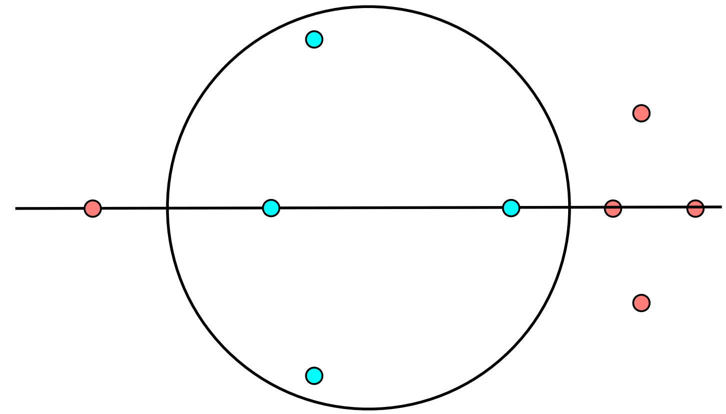
Algebraic integers

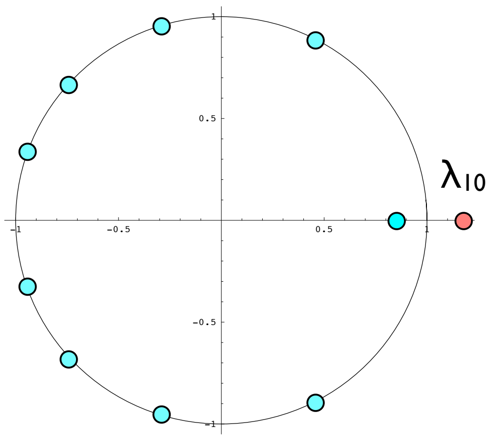
What is the smallest integer λ > 1?



Mahler measure

M(λ) = product of conjugates with $|\lambda_i| > 1$

Lehmer's Number



$\lambda_{10} = 1.176280...$

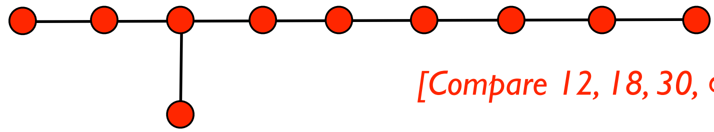
$P_{10}(x) = x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$

Smallest Salem Numbers, by Degree

		$P_d(x)$
λ_2	2.61803398	x^2-3x+1
λ_4	1.72208380	$x^4-x^3-x^2-x+1$
λ_6	1.40126836	$x^6-x^4-x^3-x^2+1$
λ_8	1.28063815	$x^8-x^5-x^4-x^3+1$
→ λ_{10}	1.17628081	$x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$
λ_{12}	1.24072642	$x^{12}-x^{11}+x^{10}-x^9-x^6-x^3+x^2-x+1$
λ_{14}	1.20002652	$x^{14}-x^{11}-x^{10}+x^7-x^4-x^3+1$

Conjecture (Lehmer) λ_{10} = inf M(α) over all algebraic integers with M(α)>0.

Lehmer's polynomial = det(xI-w) for E_{10}

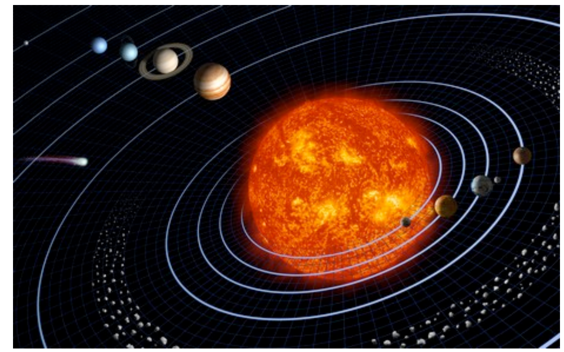


[Compare 12, 18, 30, ∞]

Theorem. *The spectral radius of any w in any Coxeter group satisfies* $r(w) = 1$ *or* $r(w) \geq \lambda_{10} > 1$.

Proof: uses Hilbert metric on the Tits cone.

Dynamics



$f : X \to X$

What is the simplest interesting dynamical system?

Entropy

Entropy of English = h = about log 3 (or less)

Schneier, Applied Cryptography, 1996

Number of possible English books with N characters is about 3^N (not 26^N)

X compact, $f : X \to X$ continuous

$h(f) = \log \lambda \Leftrightarrow$

$|\{\text{orbit patterns of length } N\}| \sim \lambda^N$.

Torus examples

$X = \text{torus } R^n/Z^n$

$f : X \to X$ linear map induced by A in $GL_n(Z)$

$h(f) = \log$ (product of eigenvalues of A with $|\lambda| > 1$)

$= \log$ [spectral radius of $f^* \mid H^*(X)$]

Lehmer's conjecture ⇔

$h(f) \geq \log \lambda_{10}$ if $h(f)>0$.

Entropy on Complex Surfaces

X = compact complex manifold, dim = 2,
f : X → X holomorphic

What small values can h(f) assume?

(dim=1 ⇒ zero entropy)

Entropy: Kähler case

Theorem (Gromov, Yomdin) *The entropy of an automorphism f : X → X of a compact Kähler manifold is given by:*

$$h(f) = \log [\text{spectral radius of } f|H^*(X)] .$$

Cor. *For surfaces,*
$h(f) = \log$ [a Salem number] $= \log \rho(f|H^2(X))$.

Complex Surfaces

Theorem (Cantat) *A surface X admits an automorphism f : X → X with positive entropy only if X is birational to:*

- 6 — *a complex torus $\mathbb{C}^2/\Lambda$,*
- 22 — *a K3 surface*, or* (*or Enriques)
- ∞ — *the projective plane $\mathbb{P}^2$.*

Synthesis Problem:
Salem number ⇒ surface + map

Complex torus $\mathbb{C}^2/\Lambda$

Theorem. *There exists a 2D complex torus automorphism with* $h(f) = \log(\lambda_6)$.

(minimum possible for 2D torus)

Complement. *For a projective torus, one can achieve* $h(f) = \log(\lambda_4)$ *and this is optimal.* *(1.722 > 1.401)*

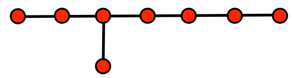
Synthesis: $f|H^1(X,\mathbb{Z}) \Rightarrow \Lambda \subset \mathbb{C}^2 \Rightarrow X$

Rational Surfaces

X = blowup of $\mathbb{P}^2$ at n points

$H^2(X,\mathbb{Z}) \simeq \mathbb{Z}^{1,n} \supset K_X^{\perp} \simeq$ [E_n lattice]

$K_X = (-3,1,1,\ldots,1)$



$\mathrm{Aut}(X) \subset W(E_n)$ *(Nagata)*

Theorem. *The Coxeter automorphism of E_n can be realized by an automorphism $F_n : X_n \to X_n$ of $\mathbb{P}^2$ blown up at n suitable points.*

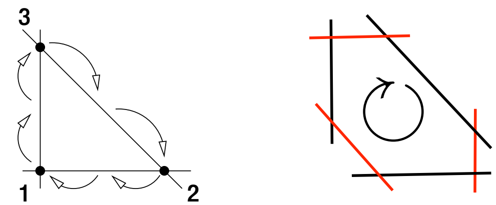
E_n spherical $\Leftrightarrow$ F_n periodic $\Leftrightarrow$ $n \leq 8$

(Kantor, 1890s)

Example: $F_3(x,y) = (y,y/x)$

$(x,y) \to (y,y/x) \to (y/x,1/x) \to (1/x,1/y) \to (1/y,x/y) \to (x/y,x) \to (x,y)$

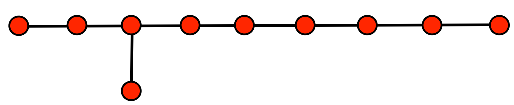
$\mathbb{P}^2$



X_3

Lehmer's automorphism

$F_{10} : X_{10} \to X_{10}$



First case where $h(F_n) > 0$

Theorem. The map F_{10} has minimal positive entropy among all surface automorphisms, namely $h(F_{10}) = \log(\lambda_{10})$.

Rational Surfaces: Synthesis

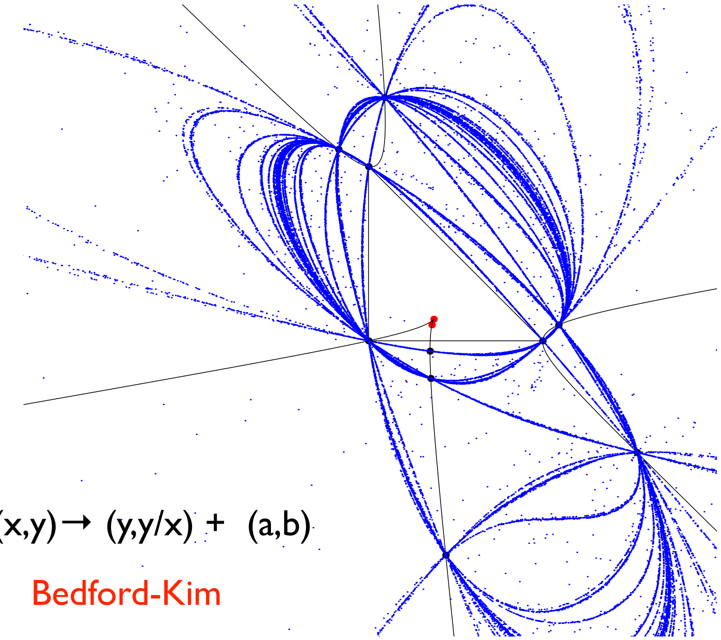
X = blowup of n points on a cuspidal cubic C in $\mathbb{P}^2$

[E_n lattice] $\simeq \mathrm{Pic}^0(X_n) \to \mathrm{Pic}^0(C) \simeq \mathbb{C}$

$\hookrightarrow$ Coxeter element w $\hookrightarrow$ Eigenvalue λ of w

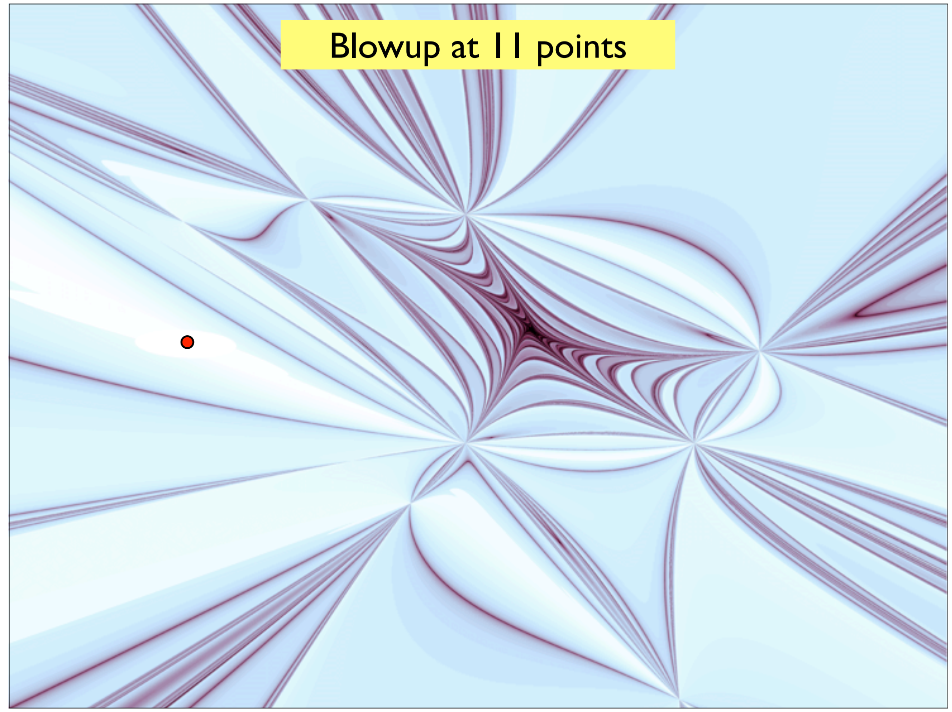
λ eigenvector of w $\Rightarrow$ positions of n points on C

10 points on a cuspidal cubic



$(x,y) \to (y, y/x) + (a,b)$

Bedford-Kim



K3 surfaces/$\mathbb{R}$

$X \subset \mathbb{R}^3$ defined by

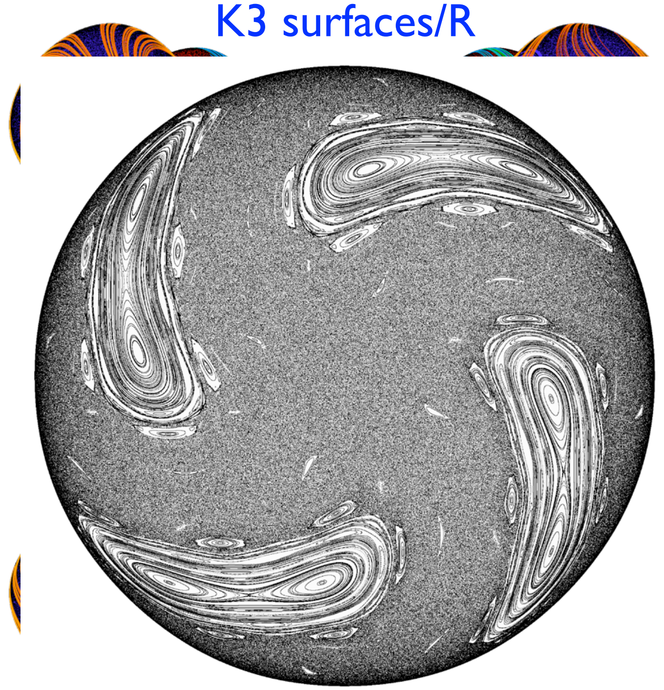
$$(1+x^2)(1+y^2)(1+z^2) + Axyz = 2$$

$f : X \to X$ defined by

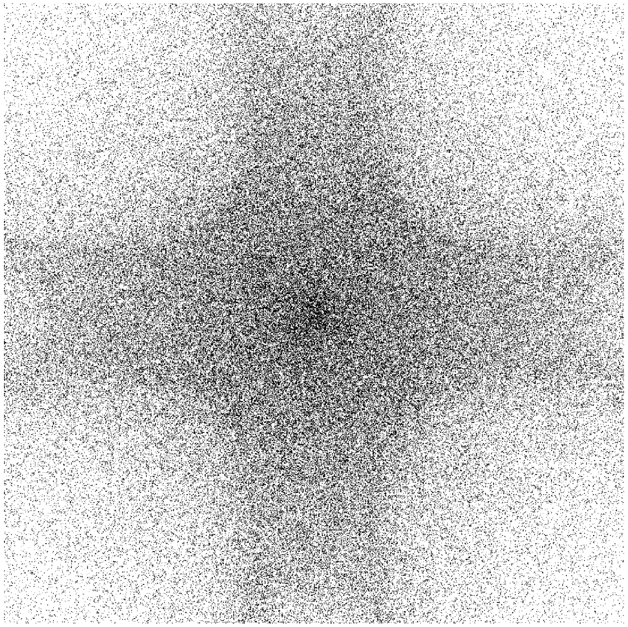
$$f = I_x \circ I_y \circ I_z$$

The map f is area-preserving!

K3 surfaces/R



Complex Orbit



K3 surfaces: Glue results

Theorem. *There exists a K3 surface automorphism with* h(f) = log(λ_{10}), *and this is optimal.*

(minimum possible)

(Oguiso - λ_{14} = 1.2002)

Complement. *For projective K3 surfaces, one can achieve* h(f) = log(λ_6).

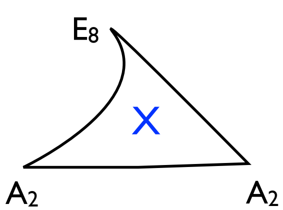
(1.401 > 1.176)

Synthesis of a K3 automorphism with entropy log λ_{10}

Coxeter automorphism of $E_{10}(a)$ $(3,7)$ — $\mathbb{F}_3^2$ — Positive automorphism of $A_2 \oplus A_2$ $(0,4)$

Positive automorphism of $A_2 \oplus A_2$ — Identity factor E_8 $(0,8)$

$H^{2,0}+H^{0,2}$
+transcendental cycles



NS(X)
Signature (0,12)
determinant 9
blows down to 3 points

Assembly of a projective K3 automorphism with entropy log λ_6

Salem factor $(1,5)$ — $\mathbb{F}_2^2$ — Coxeter automorphism of $A_2(2)$ $(0,2)$

Salem factor — $\mathbb{F}_3^6$ — Coxeter automorphism of $A_{12}(a)$ $(2,10)$

Coxeter automorphism of $A_2(2)$ — $\mathbb{F}_3$ — Identity factor $(0,2)$

Coxeter automorphism of $A_{12}(a)$ — $\mathbb{F}_{13}$ — Identity factor $(0,2)$

$H^{2,0}+H^{0,2}$
+transcendental cycles

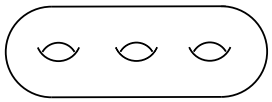
NS(X): signature (1,9), determinant 9477 = $3^6 \cdot 13$

Problem: Find this example in projective space!

$\mathcal{M}_g =$

Moduli space

of Riemann surfaces X of genus g

= {  }

-- a complex variety, dimension 3g-3

Teichmüller metric: every holomorphic map

$$f : \mathbb{H}^2 \to \mathcal{M}_g$$

is distance-decreasing.

Entropy on topological surfaces

$\mathrm{Mod}_g = \{f : \Sigma_g \to \Sigma_g\}/\text{isotopy} = \pi_1(\mathcal{M}_g)$

$h(f) = \min \{h(g) : g \text{ isotopic to } f\}$

= length of loop on moduli space represented by [f].

$\geq$ log spectral radius of f^* on $H^1(\Sigma_g)$

Stringlike Synthesis: f ⇒ a loop of Riemann surfaces

Conjectures on finite covers

$$\begin{array}{ccc} \Sigma_h & \xrightarrow{F} & \Sigma_h \\ \downarrow & & \downarrow \\ \Sigma_g & \xrightarrow[f]{} & \Sigma_g \end{array}$$

h(f) = sup log spectral radius of F^* on $H^1(\Sigma_h)$, over all finite covers.

⇕

The super period map $\mathcal{M}_g \to \prod_C \mathcal{A}_h$ is an isometry.

(from the Teichmüller metric to the Kobayashi metric)

Kazhdan's Theorem

$$\begin{array}{l} \mathbb{H} \\ \downarrow \\ Y \to \mathrm{Jac}(Y) = \mathbb{C}^h/\Lambda \\ \downarrow \\ X \end{array}$$

The hyperbolic metric on X is the limit of the metrics inherited from the Jacobians of finite covers of X.

Counterexamples

Theorem.
The super period map

$$\mathcal{M}_g \to \prod_{\mathcal{C}} \mathcal{A}_h$$

is not an isometry in the directions coming from quadratic differentials with odd order zeros.

Corollary.
The entropy of most mapping classes
$f : \Sigma_g \to \Sigma_g$
cannot be detected homologically, even after passing to finite covers.

⟨Spectral Gap⟩

Surface maps with genus → ∞

(after Lanneau-Thiffeault, E. Hironaka, Farb-Leininger-Margalit)

Let δ_g = exp (length of shortest geodesic on $\mathcal{M}_g$.)

Known: $\delta_1 = (3+\sqrt{5})/2 \Leftrightarrow$ matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

$= \lambda_2$ = smallest Salem number of degree 2

$\delta_2 = \lambda_4$ = smallest Salem number of degree 4

$\delta_g = 1 + O(1/g)$ *(not Salem for g >> 0!)*

Conjecture: $\lim\, (\delta_g)^g = \delta_1 = (3+\sqrt{5})/2 = \lambda_2$

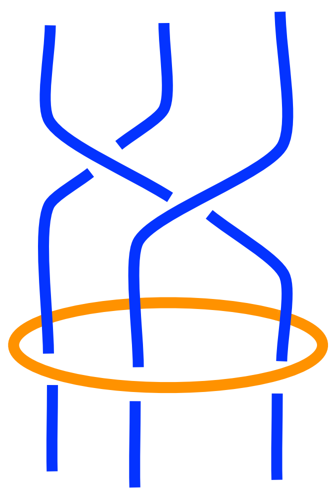
A link L from the shortest loop in the moduli space of genus one

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

$$= \mathrm{Mod}_1 \cong \mathrm{Mod}_{0,4}$$

⇕

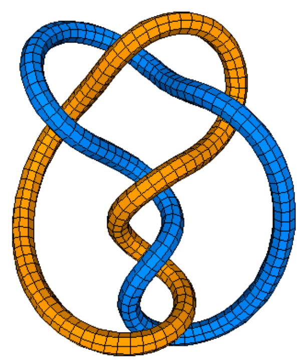
Braids on 3 strands



$M^3 = S^3$-L fibers over S^1

Other fibrations $M^3 \to S^1$

$M^3 = S^3$-L



$H_1(M)$

Teichmüller polynomial

	-1	
1	-1	1
	-1	

X^2-3X+1

$T_n(X) = X^{2n} - X^n(X+1+X^{-1}) + 1$

Teichmüller polynomial: special values

$T_2(X) = X^4 - X^3 - X^2 - X + 1$

⇒ genus 2 example with $h(f) = \log \lambda_4$

$T_6(X) = X^{12} - X^7 - X^6 - X^5 + 1$

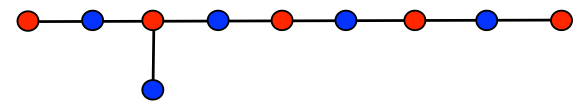
$= (X^2-X+1)(X^{10}+X^9-X^7-X^6-X^5-X^4-X^3+X+1)$

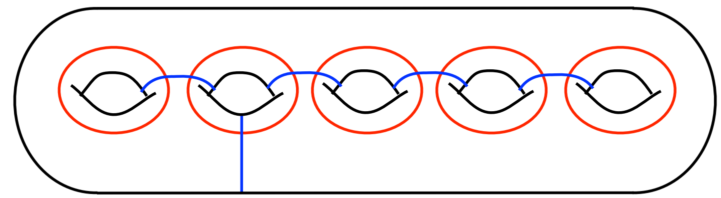
⇒ genus 5 example with $h(f) = \log \lambda_{10}$

$T_n(X)$ ⇒ genus g examples with $h(f) \sim \log \lambda_2^{1/g}$,

in agreement with conjecture

Lehmer's number for topologists

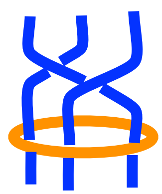




$\psi = \tau_A \tau_B$ in the mapping-class group for genus 5

$h(\psi) = \log \lambda_{10}$

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 \Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow$$



⇒ other fibrations ⇒ Lehmers number in genus 5

Lehmer's number is implicit in the golden mean.

(two shadows of the same object)