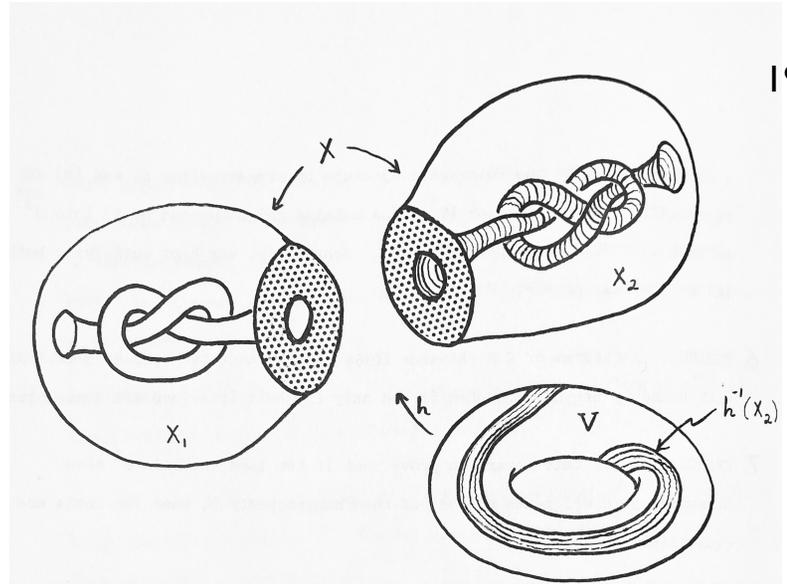


Low-dimensional mathematics

Curtis T McMullen
Harvard University

Low-dimensional topology



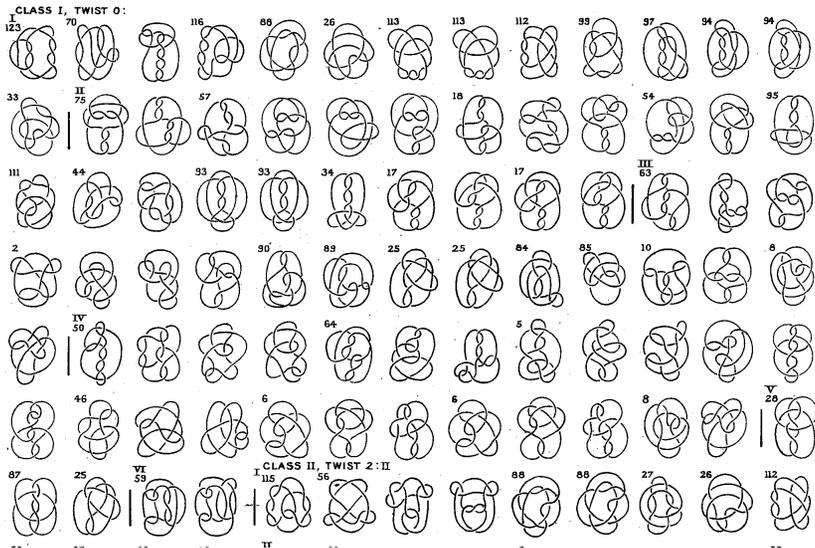
1970s

Trans. Roy. Soc. Edin.

Vol. XXXIX.

PROF. LITTLE: NON-ALTERNATE \pm KNOTS. Tait and Little 1899

PLATE I.



MSRI 1984-5



S.-S. Chern



Calvin (Cal) Moore



2223 Fulton St



August 01, 1984 - July 31, 1985

Program [Low-dimensional Topology](#)

Organizers: R. Edwards (chairman), R. Kirby, J. Morgan, W. Thurston

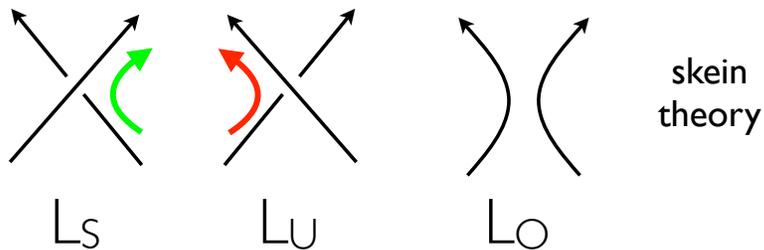
Updated On Feb 10, 2023 01:09 PM PST

August 01, 1984 - July 31, 1985

Program [K-Theory, Index Theory, and Operator Algebras](#)

Organizers: A. Connes (chairman), R. Douglas, M. Takesaki

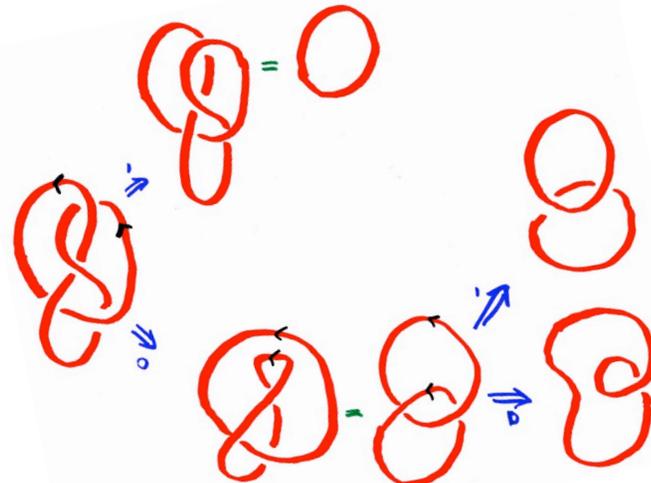
I. Jones polynomial



$$(1/t)V_S - tV_U = (t^{1/2} - t^{-1/2})V_O$$

Jones polynomial of the figure 8 knot

$$V(K, t) = t^{-2} - t^{-1} + 1 - t + t^2$$



Questions

1. Does $V(K, t) = 1$ imply K is the unknot?
2. What does $V(K, t)$ measure?
3. How hard is it to compute $V(K, t)$?

Questions



Witten

2. What does $V(K, t)$ measure?

$$\begin{aligned} \langle K \rangle &= \int \text{Tr}(\oint_K A) e^{2\pi i k \text{CS}(A)} DA \\ &= (q^{1/2} + q^{-1/2}) V(K, 1/q) \\ q &= \exp(2\pi i/(2+k)) \end{aligned}$$

3. Can $V(K, t)$ be found quickly by a quantum computer?

Braids

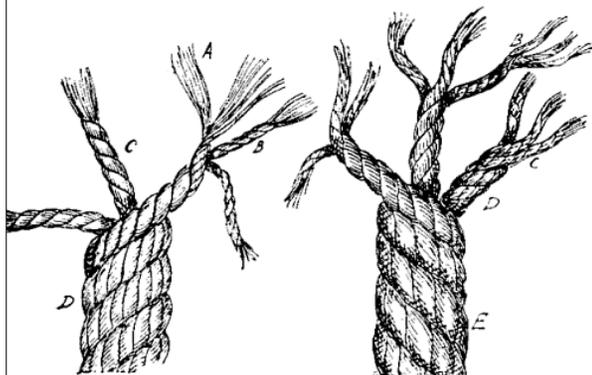
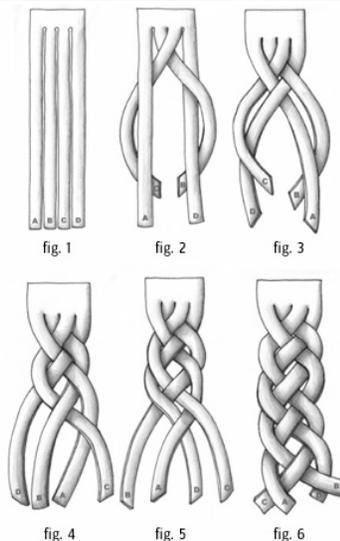


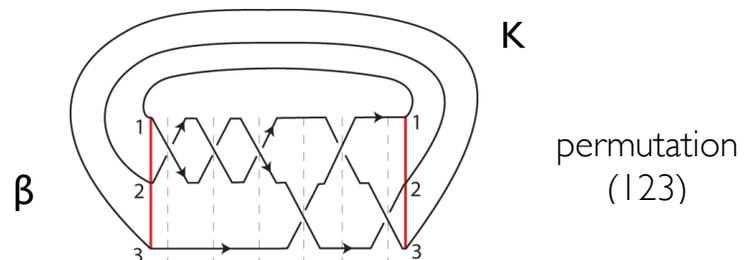
FIG. 1.—Construction of rope.



$$P_n = \left\{ \begin{array}{l} \text{Polynomials } p(x) \text{ of degree} \\ n, \text{ with distinct roots} \end{array} \right\}$$

Braid group
 $B_n = \pi_1(P_n)$

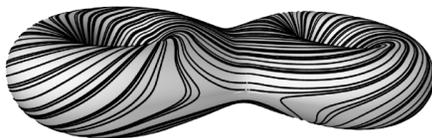
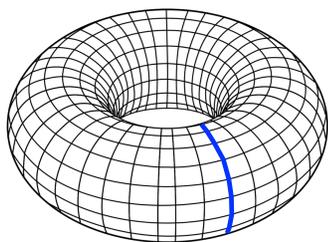
Braids as quantum permutations



$$V(K, t) = -(t^{1/2} + t^{-1/2}) t^{\deg(\beta)} \text{Tr}(\beta)$$

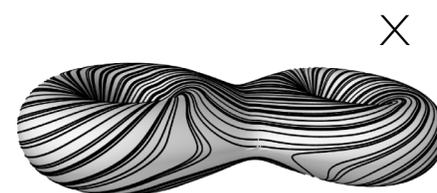
Fast computation for braid closures
 ... with few strands.

2. Knots on surfaces



Q. How to construct a random simple loop?

Hyperbolic Geometry



Closed loops, length $< L$: $\pi(L) \sim \frac{e^L}{L}$

Prime number theorem - Selberg *et al*

Simple loops, length $< L$: $\sigma(L) \sim C_X L^{6g-6}$

Mirzakhani 2008

Hyperbolic Geometry

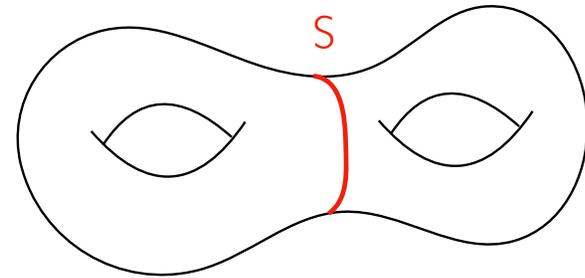


A: Choose a long simple loop at random.

Topological statistics universal

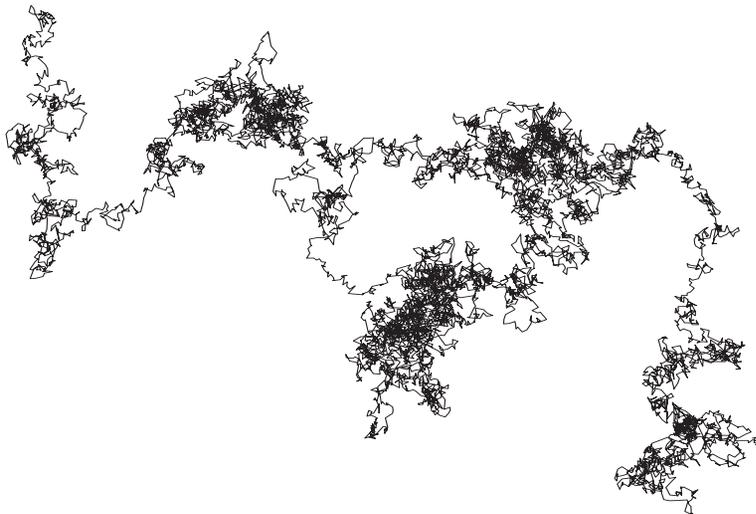
Answers are rational numbers

Example:



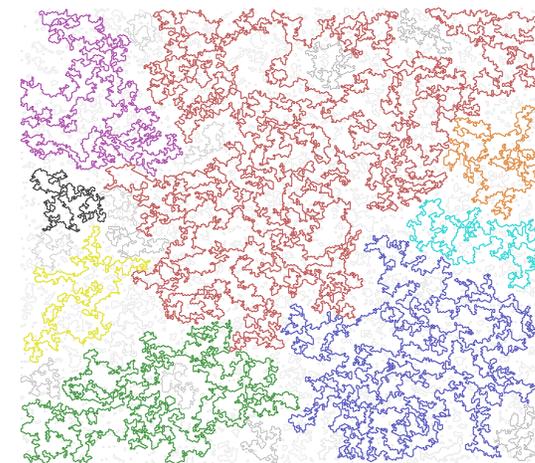
$$P(S \text{ cuts } X \text{ into 2 pieces}) = 1/49$$

Use random paths?



Brownian motion (not simple)

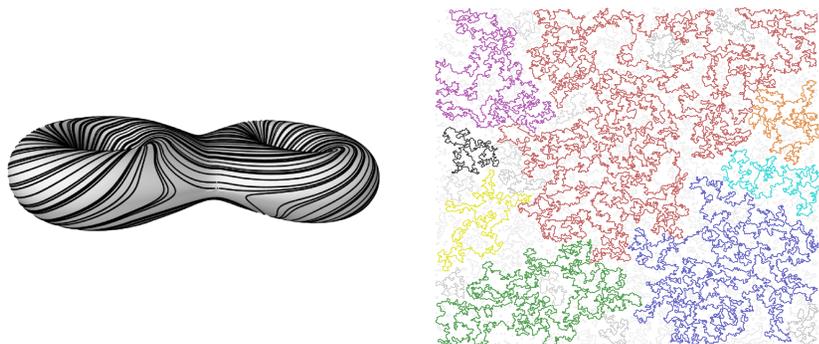
Loop soups



Conformally invariant: level sets of GFF

Lawler, Werner, Schramm, Smirnov, Sheffield, ...

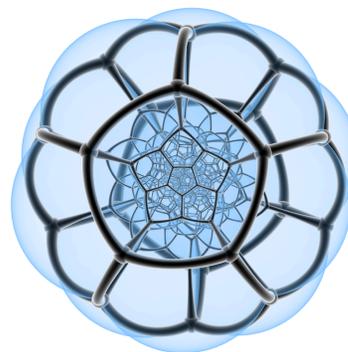
Q. Can these constructions be combined?



3. Three-manifolds

Thurston, Perelman
1983-2006

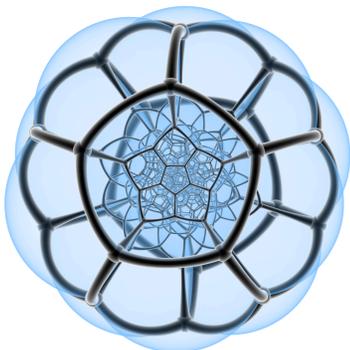
All 3-manifolds are geometric.



3. Three-manifolds

Thurston, Perelman
1983-2006

All 3-manifolds are geometric.



3. Three-manifolds

Thurston, Perelman
1983-2006

Almost all 3-manifolds are hyperbolic.



Hyperbolic 3-manifolds

$$M^3 = \mathbb{H}^3/\Gamma$$

$$\Gamma \subset \mathrm{SL}_2(\mathbb{C})$$

Mostow rigidity

Arithmetic manifolds

Ratner's theorems



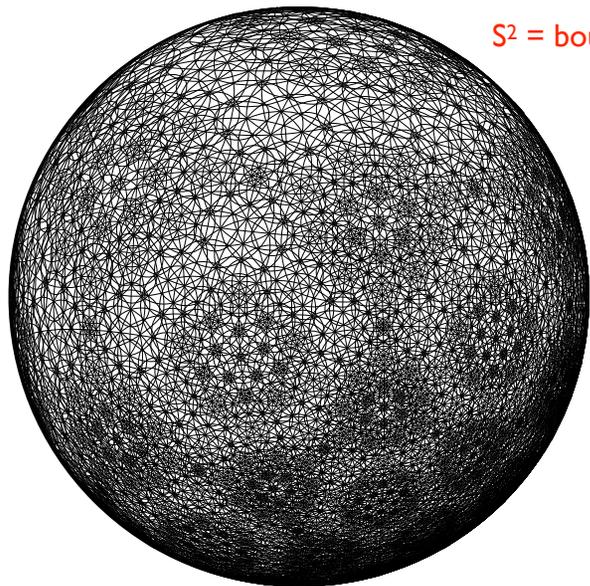
Planes in hyperbolic 3-manifolds

Given an isometric immersion

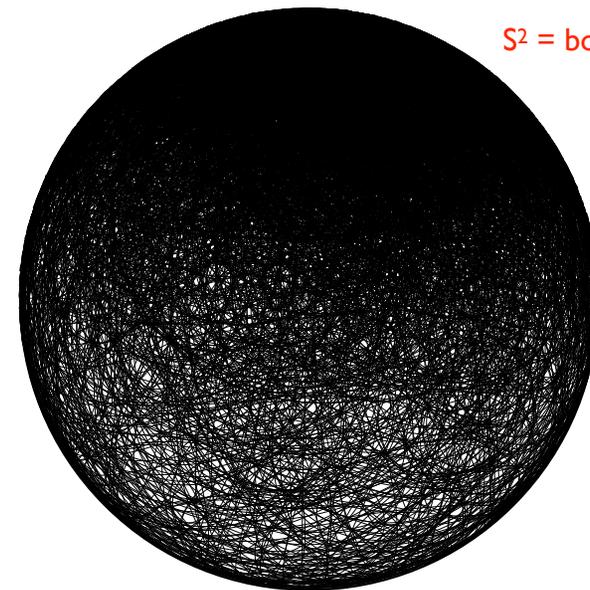
$$f: \mathbb{H}^2 \rightarrow \mathbb{H}^3/\Gamma$$

- * It can happen that $f(\mathbb{H}^2)$ is a closed surface. *(miracle)*
- * Otherwise, $f(\mathbb{H}^2)$ is dense in M .

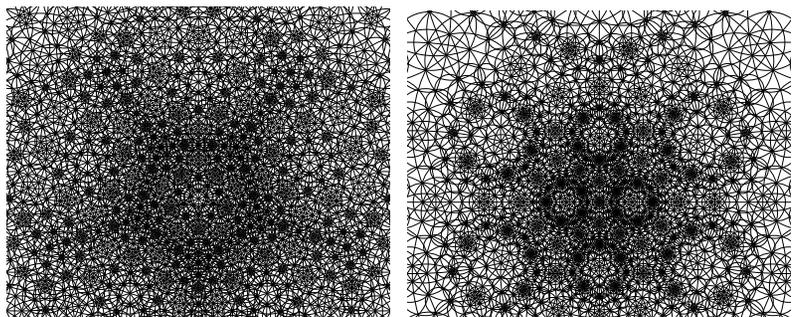
Shah, Margulis, Ratner 1991



$S^2 = \text{boundary of } \mathbb{H}^3$



$S^2 = \text{boundary of } \mathbb{H}^3$



Open problem: Do ∞ many geodesic surfaces
 $\Rightarrow M$ is arithmetic?



Yes! Margulis-Mohammadi, Fisher-Miller-Stover
 2019

Mildly warped surfaces

Kahn—Markovic 2009

Every M^3 contains a *nearly* isometrically immersed surface S .

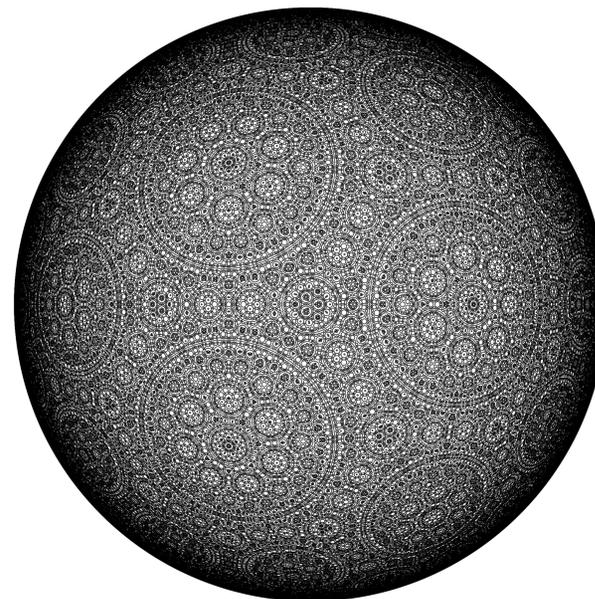
$$f: S = \mathbb{H}^2/\Delta \rightarrow M = \mathbb{H}^3/\Gamma$$

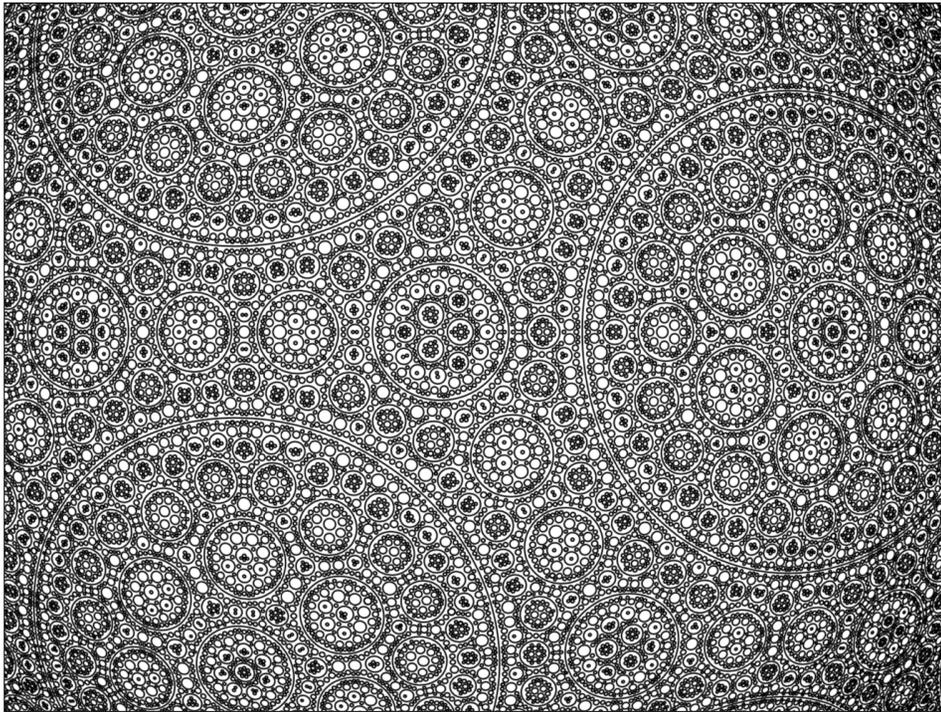
S compact, but $g(S) \gg 0$.

Topological finale

Agol—Wise 2012

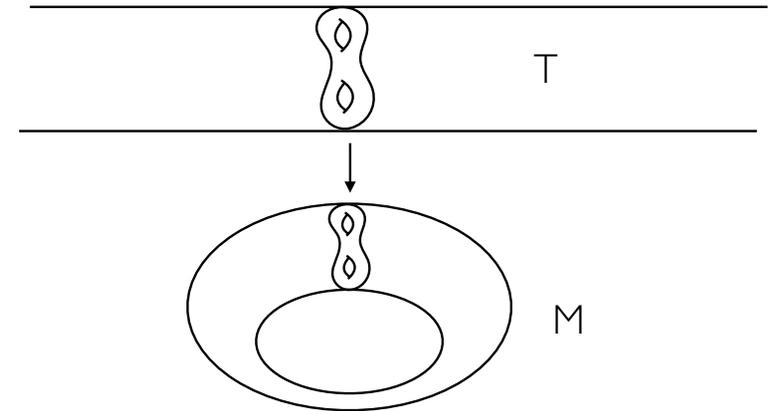
After passing to a finite cover, M^3 contains an *embedded* almost geodesic surface.





Topological finale, *bis*

M^3 has a finite cover that fibers over S^1 .



This dubious-sounding question seems to have a definite chance for a positive answer.

Thurston, 1985

4. Planes in moduli spaces

For any *complex geodesic*

$$f: \mathbb{H}^2 \rightarrow \mathcal{M}_g,$$

the closure V of its image is an algebraic variety.

Eskin, Mirzakhani, Mohammadi; Filip 2016

Q. What V are possible?

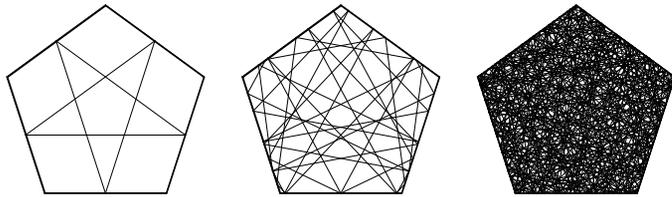
Teichmüller curves

There exist (rare jewels)

$$f: \mathbb{H}^2 \rightarrow \mathcal{M}_g,$$

such that the image V is an algebraic curve.

Veech, M, Bouw-Möller, ...



Teichmüller surfaces

There exist a complex geodesic

$$f: \mathbb{H}^2 \rightarrow \mathcal{M}_g$$

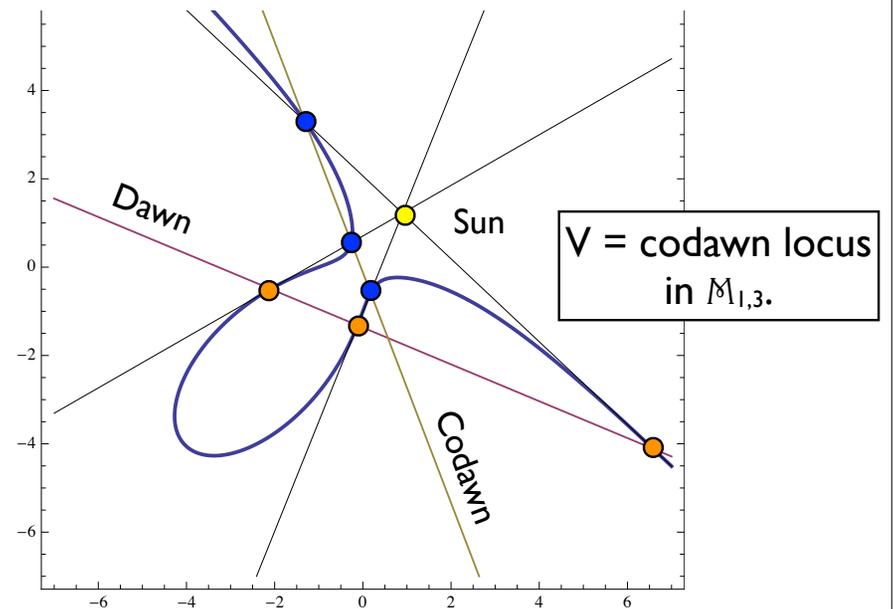
such that the image closure V is a complex, totally geodesic surface.

M, Mukamel, Wright 2016

1879

A TREATISE
ON THE
HIGHER PLANE CURVES:
INTENDED AS A SEQUEL
TO
A TREATISE ON CONIC SECTIONS.
BY
GEORGE SALMON, D.D., D.C.L., LL.D., F.R.S.,
REGIUS PROFESSOR OF DIVINITY IN THE UNIVERSITY OF DUBLIN.
THIRD EDITION.
Dublin:
HODGES, FOSTER, AND FIGGIS, GRAFTON STREET,
BOOKSELLERS TO THE UNIVERSITY.
MDCCLXXIX.

The solar configuration



Flip side of rigidity.

Q. How to test if a real number x is in $\mathbb{Q}(\sqrt{D})$?

$$x = [a_0, a_1, a_2, a_3, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

A. x is in $\mathbb{Q}(\sqrt{D})$ iff a_i 's repeat.

Arithmetic chaos?

Does the number of $[a_1, \dots, a_p]$ in $\mathbb{Q}(\sqrt{D})$ with $a_i \leq 2$ grow exponentially as the period $p \rightarrow \infty$?

Example:

$[1], [1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2], \dots?$
lie in $\mathbb{Q}(\sqrt{5})$.

Yes (mostly) for $\mathbb{Q}+\mathbb{Q}$; "Zaremba's conjecture",
Bourgain-Kontorovich 2011

5. Complex dynamics

Klein

Γ acting on $\widehat{\mathbb{C}}$

Quotient 3-manifold

Fatou and Julia

$$f: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$$

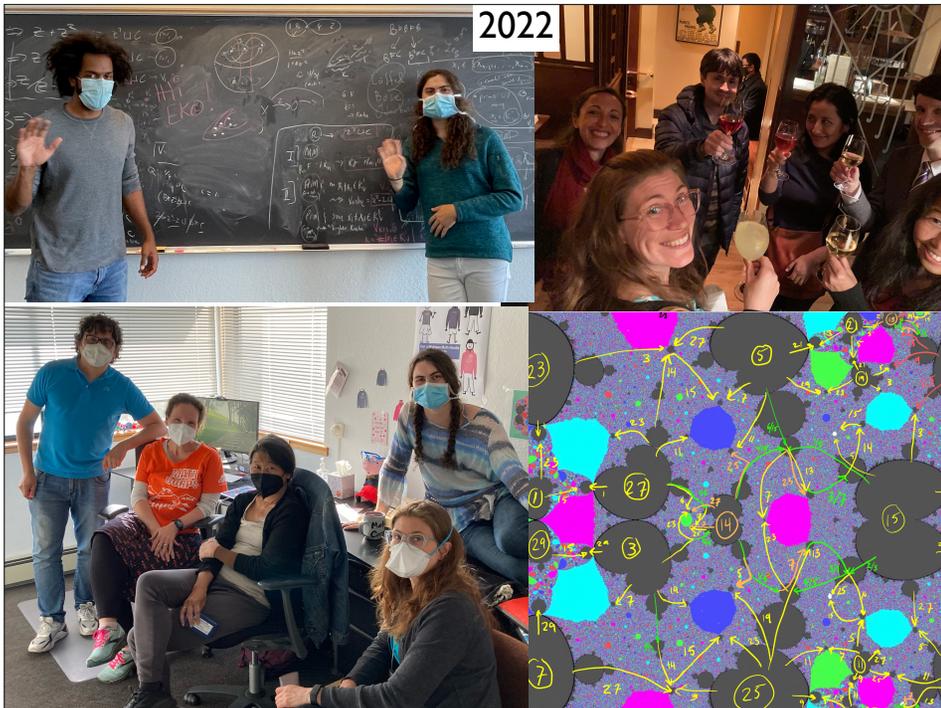
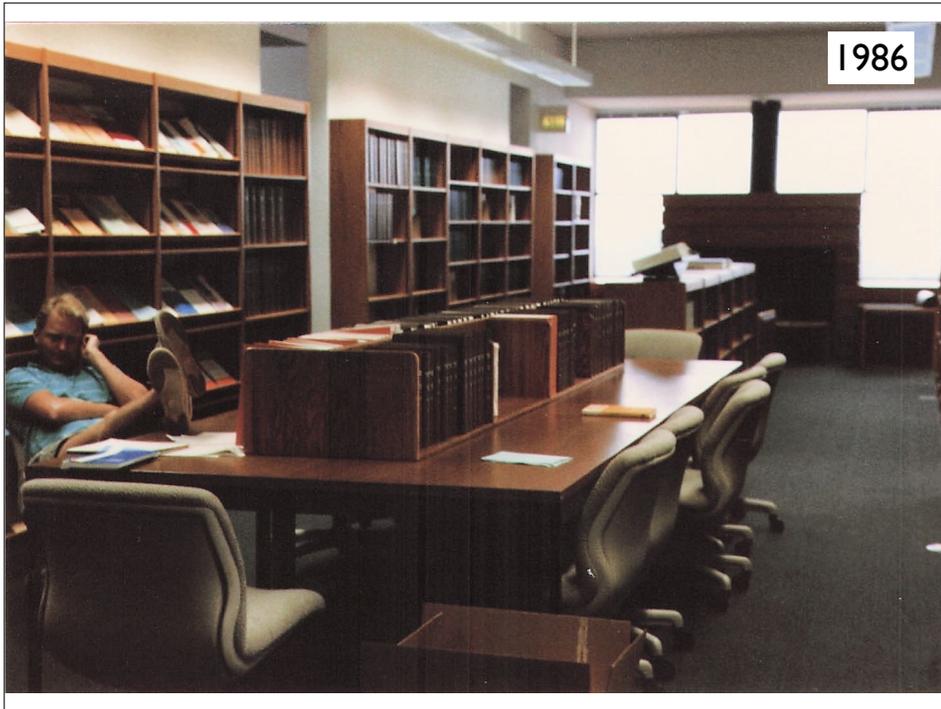
?

Complex dynamics at MSRI

1985 (with moduli/string theory)

1995 (with Kleinian groups)

2022 (with stochastic complex analysis)

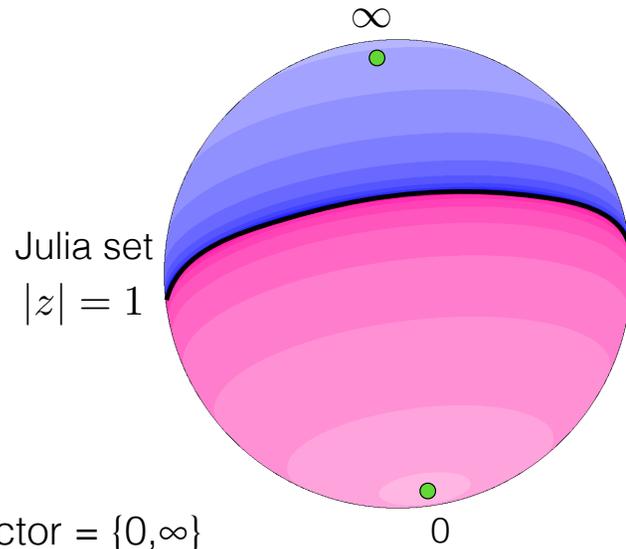


Dynamics of a typical rational map

$$f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$$

Attractor = 2 points

Dynamics of $f(z) = z^2$



The Mandelbrot set

$$f_c(z) = z^2 + c$$

$$0 \rightarrow c \rightarrow c^2 + c \rightarrow (c^2 + c)^2 + c \rightarrow \dots$$

$$M = \{c : \text{orbit of } 0 \text{ remains bounded}\}$$

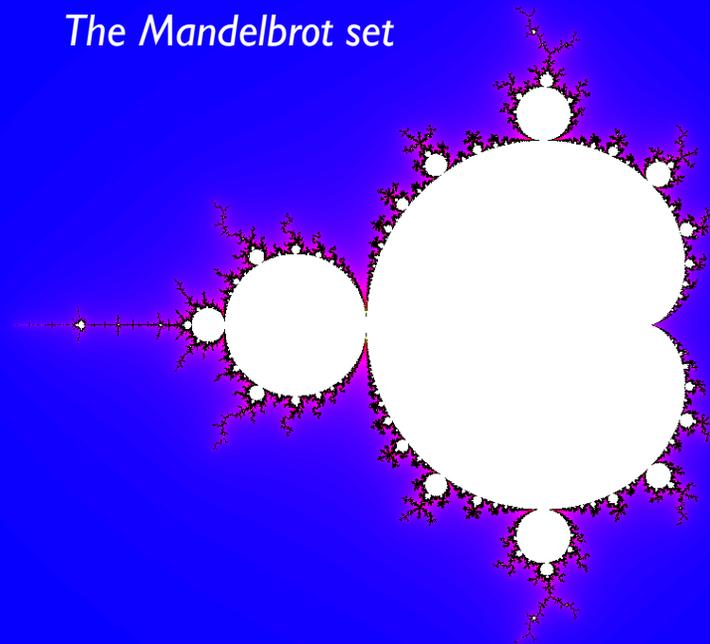
The Mandelbrot set

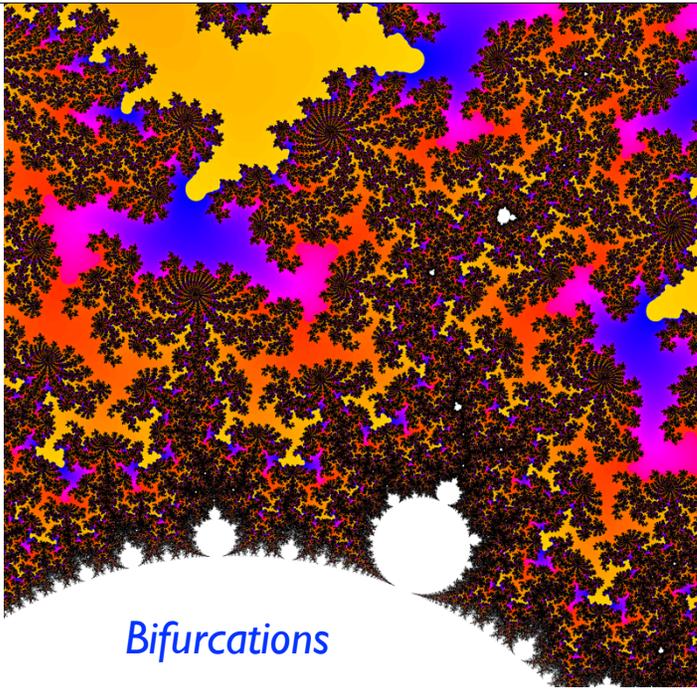
$$f_c(z) = z^2 - c$$

$$0 \rightarrow -c \rightarrow c^2 - c \rightarrow (c^2 - c)^2 - c \rightarrow \dots$$

$$M = \{-c : \text{orbit of } 0 \text{ remains bounded}\}$$

The Mandelbrot set





Bifurcations

Universality

A copy of the Mandelbrot set occurs near any bifurcation in any family of rational maps.

Dynamical Cyclotomy

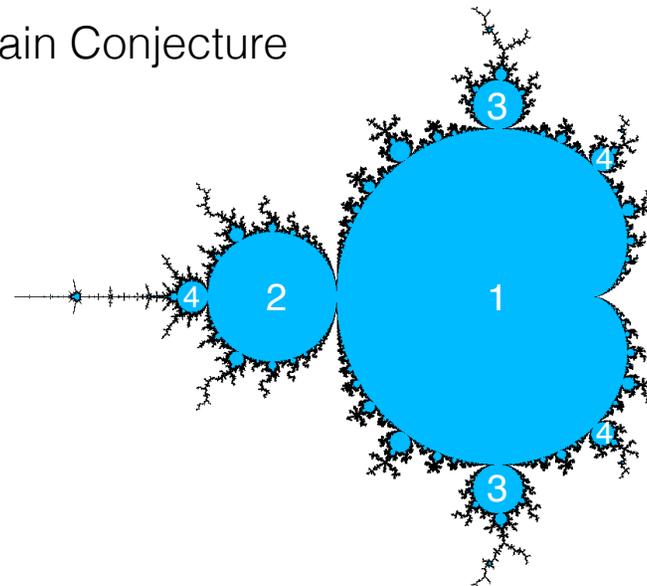
$$\Phi_n(z) = 0 \iff z \text{ has order } n \text{ in } S^1$$

$$G_n(c) = 0 \iff 0 \text{ has period } n \text{ under } f_c(z) = z^2 + c$$

$$\prod_{d|n} \Phi_d(z) = z^n - 1$$

$$\prod_{d|n} G_d(z) = f_c^n(0)$$

Main Conjecture



Every component of the interior of M contains a zero of some $G_n(c)$.

Dynamical Cyclotomy

$$G_1(c) = c$$

$$G_2(c) = c + 1$$

$$G_3(c) = c^3 + 2c^2 + c + 1$$

$$G_4(c) = c^6 + 3c^5 + 3c^4 + 3c^3 + 2c^2 + 1$$

... ..

Q. Is $G_n(c)$ irreducible?

Dynamical Curves

$\text{Per}_n = \{\text{degree 2 rational maps with one critical point of period } n\}$

Q. (Milnor) Is Per_n irreducible?

Thm (Ramadas, 2022) If G_n is irreducible over \mathbb{Q} , then Per_n is irreducible over \mathbb{C} .

6. Solving polynomials

Problem: Find a root of

$$x^d + a_1 x^{d-1} + a_2 x^{d-2} + \dots + a_d = 0.$$

What does 'find' mean?

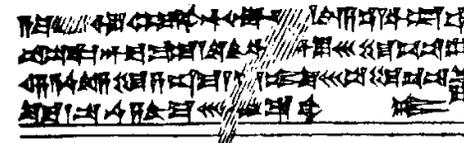
Which root?

Reliability?

Solving polynomials through the ages

Various authors

Solving the quadratic, circa 2000 BC



Solving the cubic, circa 1500 AD

REGULA.

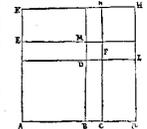
Deducto tertia parte numeri rerum ad cubum, cui addes quadratum dimidij numeri rerum, & totius accipe radicem, scilicet quadratum, quam feminabis, unij dimidium numeri quod iam in se duxeras, adijces, ab altera dimidium idem minues, habebis $\frac{2}{3}$ Binomium cum sua Apotome, inde detraha $\frac{2}{3}$ cubica Apotomae ex $\frac{2}{3}$ cubica sui Binomij, residuum quod ex hoc relinquitur, est rei estimatio. Exemplum. cubus & 6 positiones, aequatur 20, ducito 2, tertiam partem 6, ad cubum, fit 8, duc 10 dimidium numeri in se, fit 100, iunge 100 & 8, fit 108, accipe radicem quae est $\frac{2}{3}$ 108, & eam geminabis, alteri addes 10, dimidium numeri, ab altero minues tantundem, habebis Binomium $\frac{2}{3}$ 108 p: 10, & Apotomen $\frac{2}{3}$ 108 m: 10, horum accipe $\frac{2}{3}$ cubum & minue illam que est Apotomae, ab ea que est Binomij, habebis rei estimationem, $\frac{2}{3}$ v: cub: $\frac{2}{3}$ 108 p: 10 m: $\frac{2}{3}$ v: cubica $\frac{2}{3}$ 108 m: 10.

cub ³ p: 6 reb ³ æq̄lis 20	
2	20
8	10
108	
$\frac{2}{3}$ 108 p: 10	
$\frac{2}{3}$ 108 m: 10	
$\frac{2}{3}$ v: cu. $\frac{2}{3}$ 108 p: 10	
m: $\frac{2}{3}$ v: cu. $\frac{2}{3}$ 108 m: 10	

Solving the quartic, circa 1500 AD

DE MONSTRATIO.

Sic quadratum A, & dicitur in duo quadrata A d & d f, & duo supplementa d & d f, & utrum addere gnomonem x f & circūcirca, ut remaneat quadratum totum A h, dico quod talis gnomon, continebit ex duplo e & addere lineæ in e A, cum quadrato g c, nam f c constat ex e c in c r ex diffinitione data in initio secundi elementorum, et c f est æqualis c A, ex diffinitione quadrati, & per 44th primi elementorum, x f est æqualis f g, igitur duæ superficies g f & r f, continent ex o c, in duplum c A, & quadratum g c est f A, ex contrario quarta (e c) candi elementorum, igitur patet propositum, si igitur A d fit $\frac{2}{3}$ d² & c d ac d e, j quadrata, & d f g, erunt f A j quadratum, & c j 100 cessario, cum igitur uoluerimus ad d e quadrata aliqua, ad d c & d f, & fuerint c A & x A, erit ad complendum quadratum totum necessaria superficies A h g, que ut demonstratum est, constat ex quadrato g c numeri quadratorum dimidij, &



Insolvability of the quintic 1824



How to compute $\sqrt{2}$?

new guess for $\sqrt{2}$ = average of $x + 2/x$

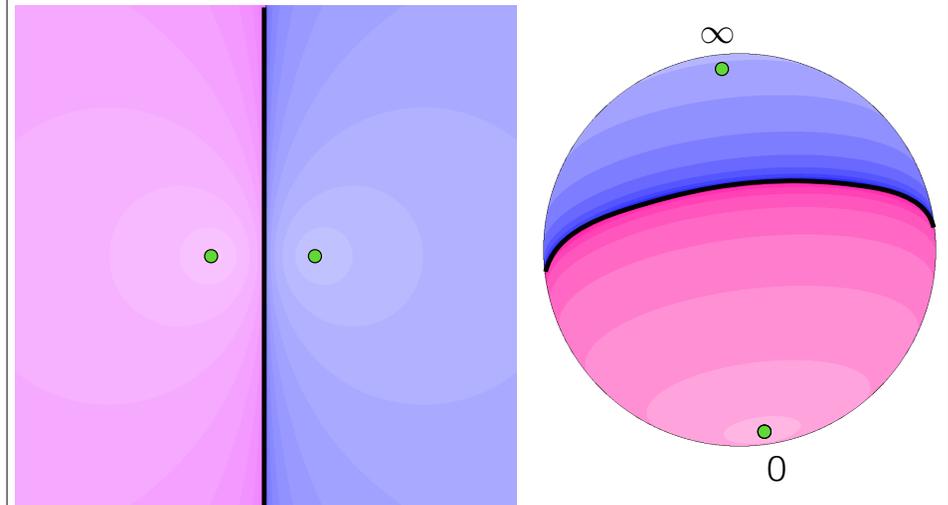
1, 1.5, 1.41666, 1.414215, 1.4142136,

Newton's method $p \rightarrow N_p(z)$

Example of a purely iterative algorithm.

(Smale)

Newton's method for $x^2 - 2$



trivial family

robust

Cayley on cubics

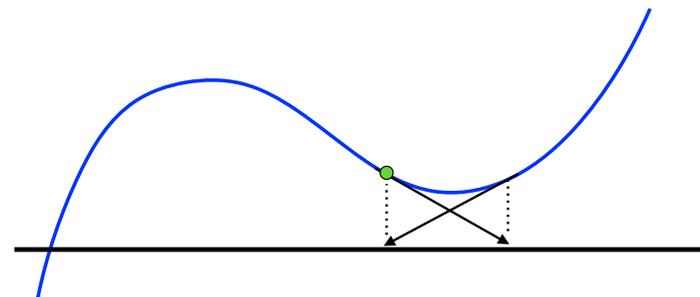
The problem is to determine the regions of the plane such that P , taken at pleasure anywhere within one region, we arrive ultimately at the point A , anywhere within another region we arrive at the point B , and so for the several points representing the root of the equation. The solution is easy and elegant for the case of a quadric equation; but the next succeeding case of a cubic equation appears to present considerable difficulty.

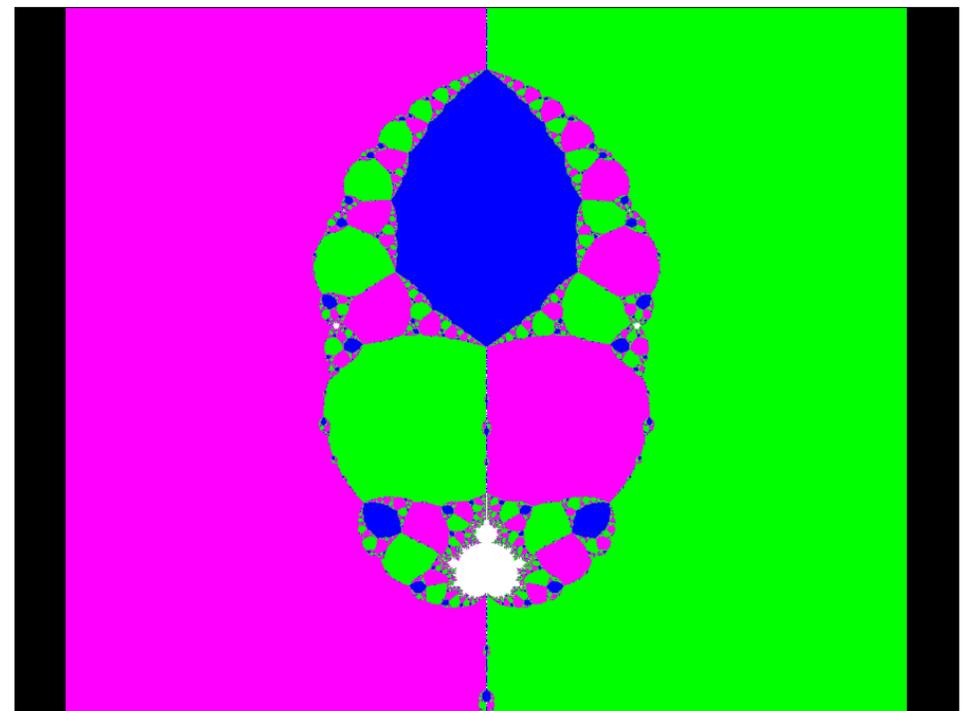
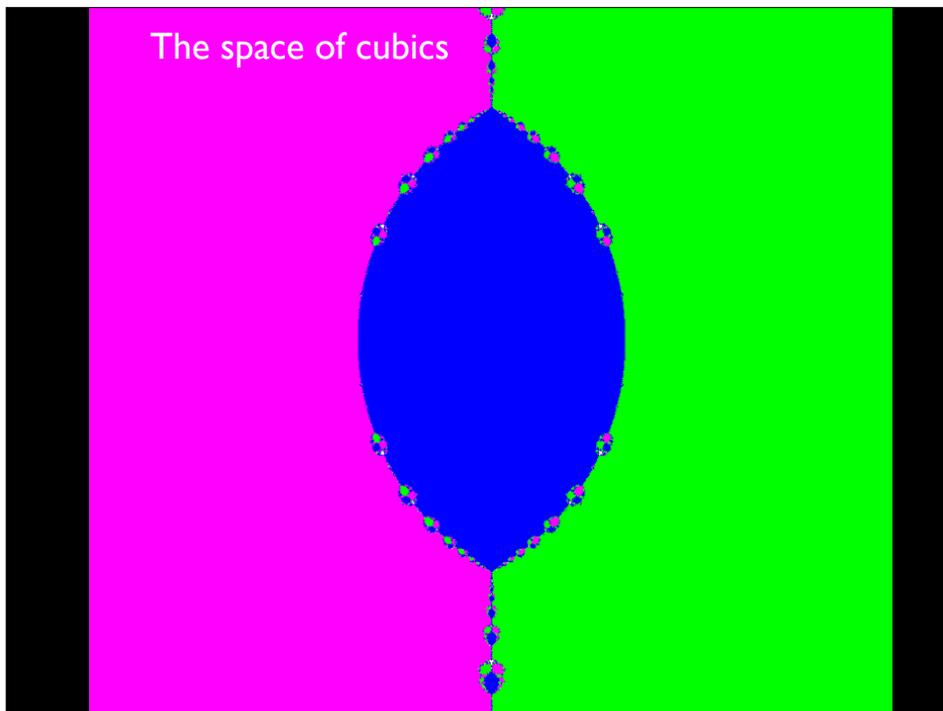
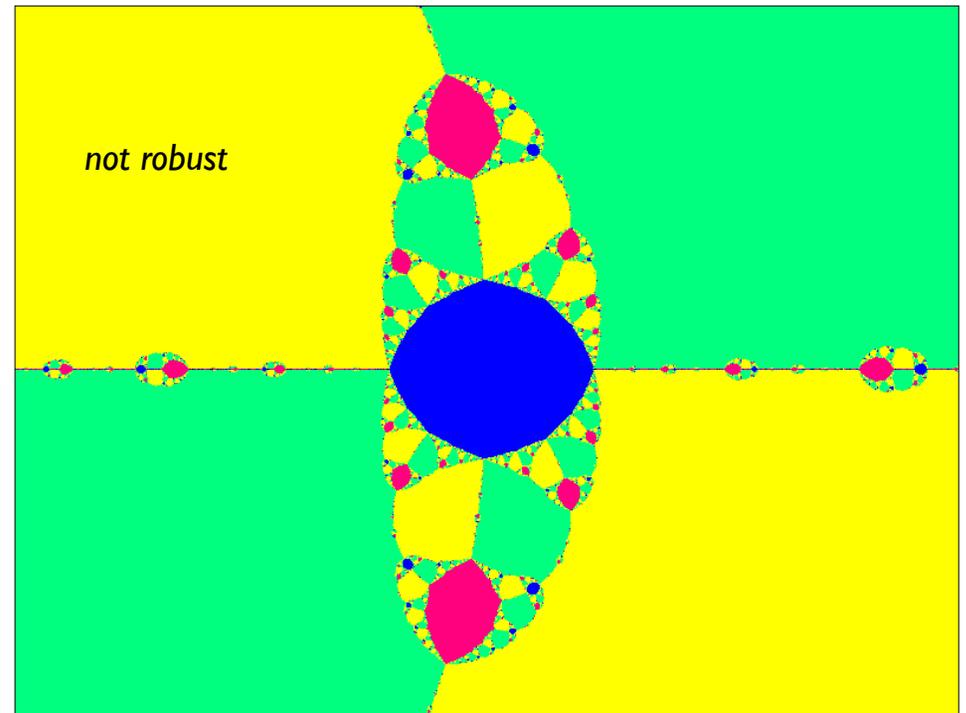
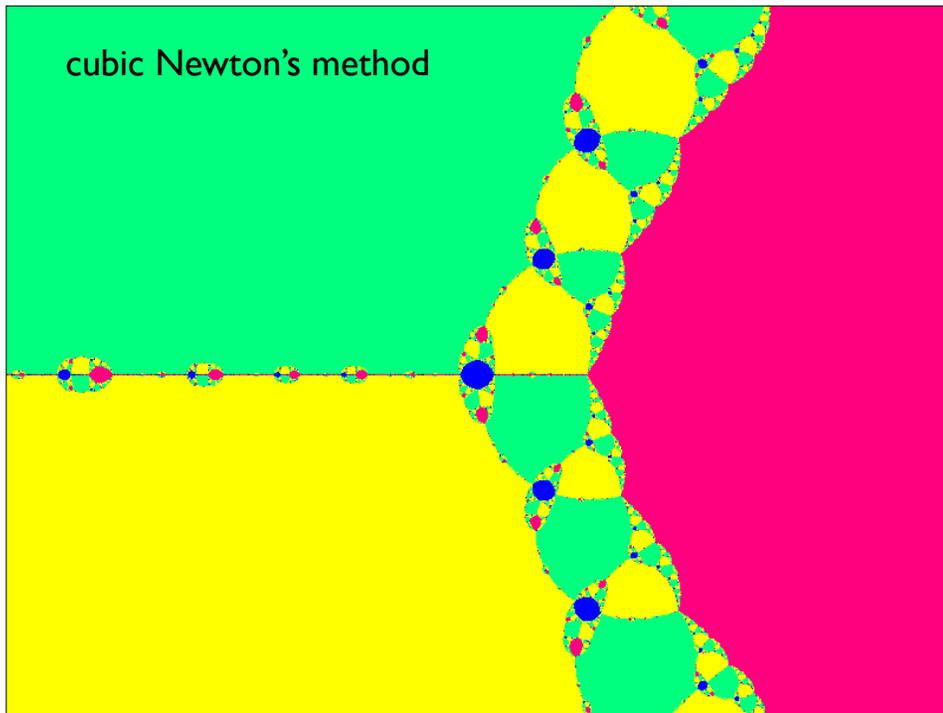


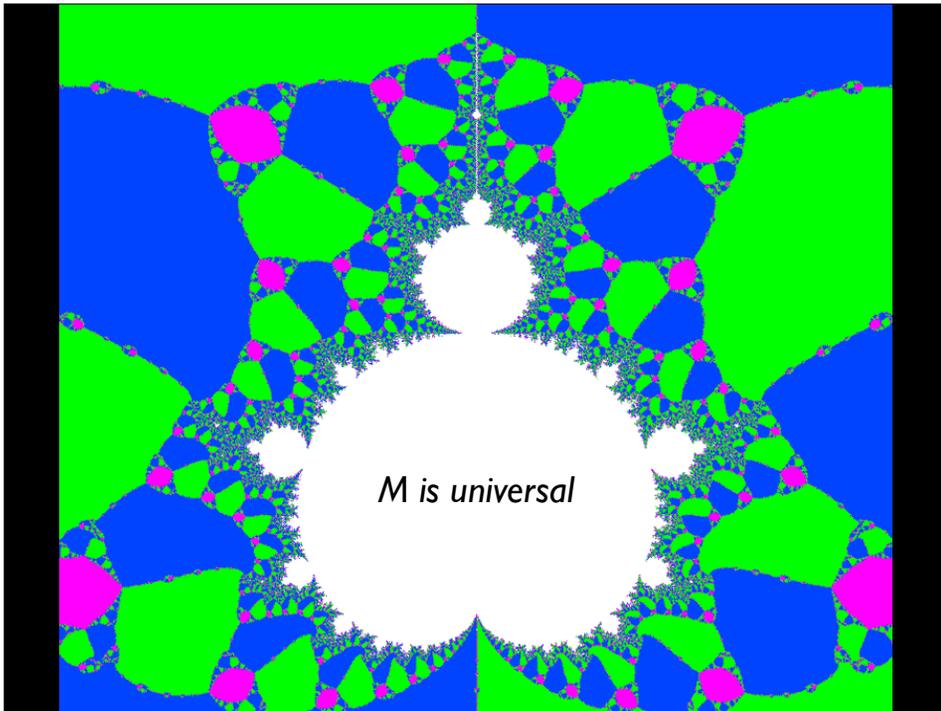
Arthur Cayley (August 16, 1821 - January 26, 1895)

from Sternberg, 2010

Newton's method for cubics







Unsolvability of the quartic

There is no robust, purely iterative algorithm to solve polynomials of degree 4 or more.

Location of failures

Every algorithm fails somewhere along this loop in the space of degree 4 polynomials

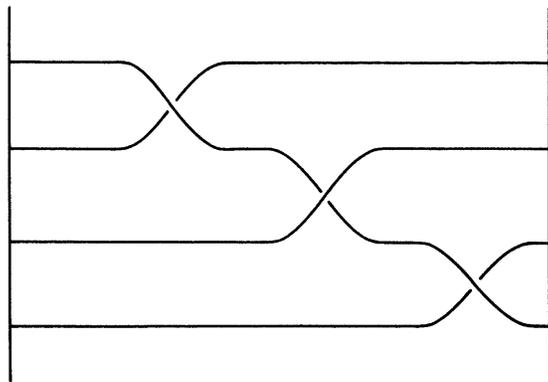
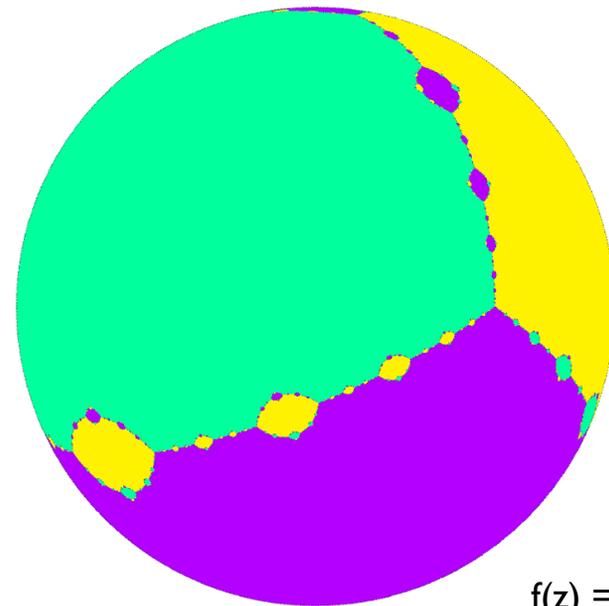


Fig. 2.1. This braid does not arise for rational maps

What about cubics?



S_3 symmetry

$$f(z) = (z^4 + 2z)/(2z^3 + 1)$$

How to solve cubics

II. There exists a unique degree 4 superconvergent algorithm for cubics. If the cubic polynomial p is given by

$$p(X) = X^3 + aX + b$$

then the algorithm is given by

$$T_p(X) = X - \frac{(X^3 + aX + b)(3aX^2 + 9bX - a^2)}{(3aX^4 + 18bX^3 - 6a^2X^2 - 6abX - 9b^2 - a^3)}.$$

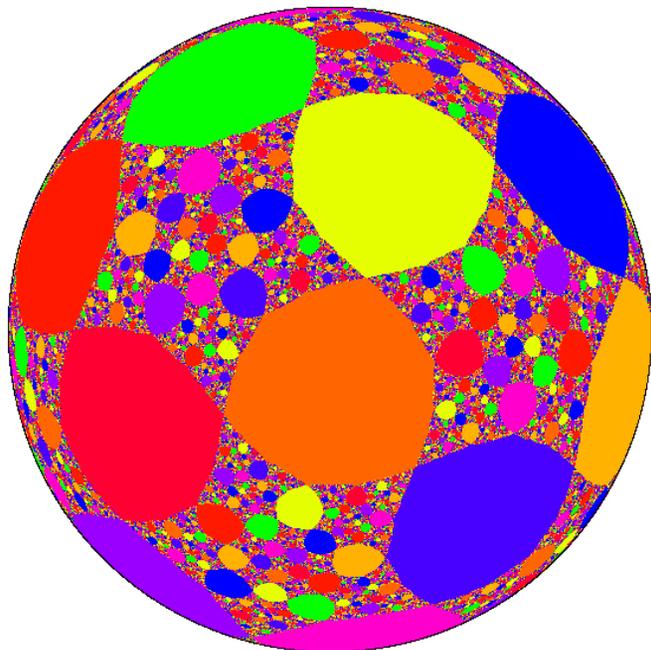
= (Tate) Newton for

$$q(X) = \frac{p(X)}{(3aX^2 + 9bx - a^2)}$$

Solvability of the quintic

The quintic can be solved by a tower of iterative algorithms.

But not the sextic, since there is no rational map with A_6 symmetry.



deg 11

Hilbert's 13th problem

(radicals $\sqrt[n]{z}$)

\subset (iterative $\lim f^n(z)$)

\subset (algebraic functions of 1 variable, $P^{-1}(z)$)

Q. Can sextic polynomials be solved using algebraic functions of 1 variable?

Arnold, Hironaka, Farb, Wolfson, ...

August 01, 1984 - July 31, 1985

Program: [Low-dimensional Topology](#)

Organizers: R. Edwards (chairman), R. Kirby, J. Morgan, W. Thurston

Updated On: Feb 16, 2023 01:00 PM PST

August 01, 1984 - July 31, 1985

Program: [K-Theory, Index Theory, and Operator Algebras](#)

Organizers: A. Connes (chairman), R. Douglas, M. Tenešek

Updated On: Feb 16, 2023 01:00 PM PST

January 01, 1988 - July 31, 1988

Program: [Geometric Function Theory](#)

Organizers: D. Drasin, F. Gehring (chairman), I. Kra, A. Meier

Updated On: Feb 16, 2023 01:00 PM PST

August 01, 1985 - July 31, 1985

Program: [Computational Complexity](#)

Organizers: R. Graham, R. Karp (co-chairman), S. Smoláček (co-chairman)

Updated On: Feb 16, 2023 06:23 PM PST

August 01, 1985 - July 31, 1988

Program: [Combinatorial Group Theory and Geometry](#)

Organizers: S. Adjan, K. Brown, S. Gersten, J. Stallings (chairman)

Updated On: Feb 16, 2023 11:41 AM PST

August 01, 1981 - July 31, 1982

Program: [Lie Groups and Ergodic Theory with Applications to Number Theory and Geometry](#)

Organizers: H. Furstenberg, M. Ramezani, P. Samuels, R. Zimmer (chairman)

Updated On: Feb 16, 2023 11:17 AM PST

January 01, 1988 - July 31, 1988

Program: [Complex Dynamics and Hyperbolic Geometry](#)

Organizers: Basil Branner, Steve Kerckhoff, Michael Lyubich, Curt McMullen (chair), and John Smillie

Updated On: Feb 16, 2023 11:17 AM PST

August 12, 2002 - December 20, 2002

Program: [Quantum Computation](#)

Organizers: Dorit Aharonov, Charles Bennett, Richard Jozsa, Yuri Manin, Peter Shor, and Umesh Vazirani (chair)

Quantum computation is an intellectually challenging and exciting area that touches on the foundations of both computer science and quantum physics.

Updated On: Jan 18, 2023 09:01 AM PST

August 20, 2007 - December 14, 2007

Program: [Technicolor Theory and Kleinian Groups](#)

Organizers: Jeffrey Brock, Richard Canary, Howard Masur, Maryam Mirzakhani, Alan Reid

These fields have each seen recent dramatic changes: new techniques developed, major conjectures solved, and new directions and connections forged. Yet progress has been made in parallel without the level of communication across these two fields that is warranted. This program will address the need to strengthen connections between these two fields, and propose new directions for each.

Updated On: Jan 31, 2023 11:01 AM PST

January 09, 2012 - May 18, 2012

Program: [Random Spatial Processes](#)

Organizers: Mikko Boulikas-Mitsu (Université de Bordeaux I, France), Richard Kenyon (Brown University), Greg Lawler (University of Chicago), Andrei Okounkov (Columbia University), and Yuval Peres (Microsoft Research Laboratories)

January 11, 2016 - May 20, 2016

Program: [Differential Geometry](#)

Organizers: Tobias Chilling (Massachusetts Institute of Technology), Simon Donaldson (State University of New York, Stony Brook), John Loft (University of California, Berkeley), Melissa Meurer (Purdue University), Gang Tian (Princeton University), [ICTP](#), [AIF](#), [Vilnius University](#) (University of Wisconsin-Madison)

January 18, 2022 - May 27, 2022

Program: [Complex Dynamics: from special families to natural generalizations in one and several variables](#)

Organizers: [LEIC](#), Sarah Koch (University of Michigan), Jaume Riera (Institut de Matemàtiques de Bordeaux), Dirk Schachler (University of Aix-Marseille (AMU)), Mitsuru Shimokura (Kyoto University), Dylan Thurston (Indiana University)

The future of (low dimensional) mathematics



It involves computers.

