Low-dimensional mathematics

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Low-dimensional topology
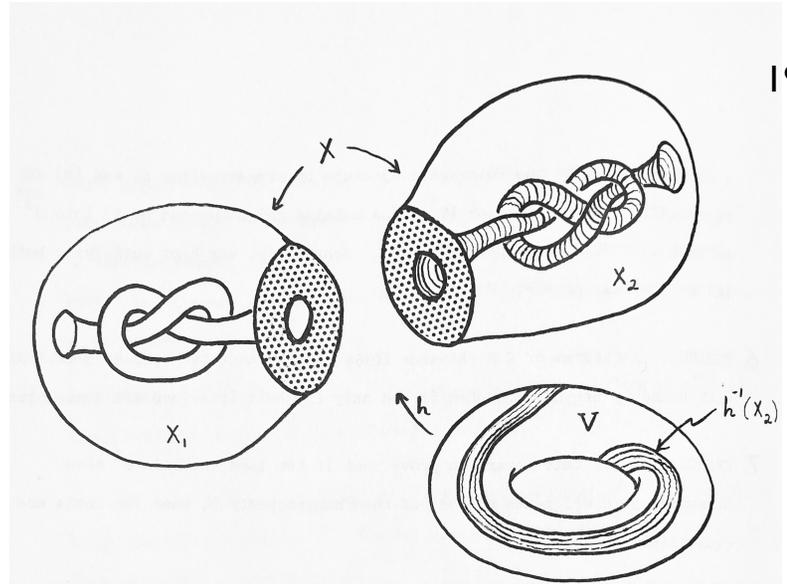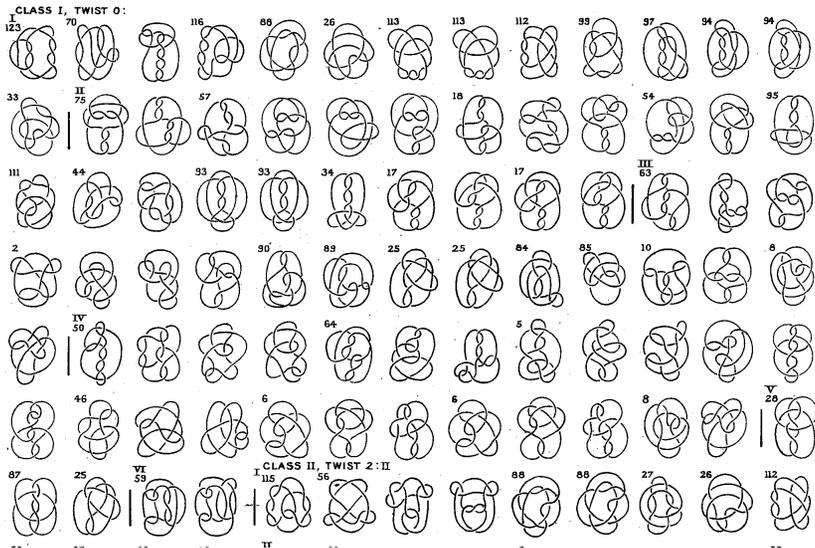
1970s

PROF. LITTLE: NON-ALTERNATE ± KNOTS.
PLATE I.

Tait and Little 1899

MSRI 1984-5

S.-S. Chern   Calvin (Cal) Moore

2223 Fulton St

August 01, 1984 - July 31, 1985

Program   Low-dimensional Topology

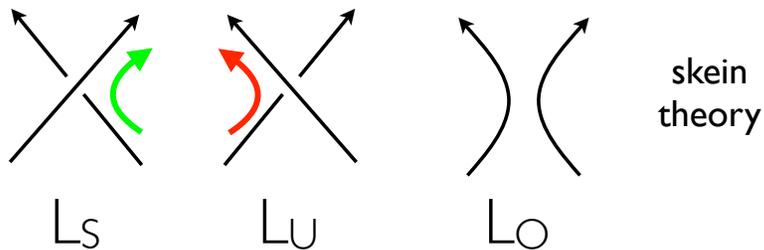*Organizers: R. Edwards (chairman), R. Kirby, J. Morgan, W. Thurston*

Updated On Feb 10, 2023 01:09 PM PST

August 01, 1984 - July 31, 1985

Program   K-Theory, Index Theory, and Operator Algebras

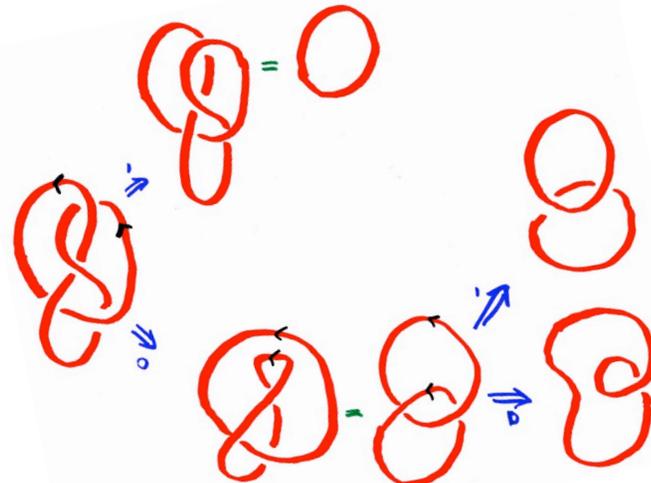*Organizers: A. Connes (chairman), R. Douglas, M. Takesaki*

1. Jones polynomial

$$L_S \qquad L_U \qquad L_O$$

skein theory

$$(1/t)V_S - tV_U = (t^{1/2} - t^{-1/2})V_O$$

Jones polynomial of the figure 8 knot

$$V(K, t) = t^{-2} - t^{-1} + 1 - t + t^2$$

Questions

1. Does V(K,t) = 1 imply K is the unknot?

2. What does V(K,t) measure?

3. How hard is it to compute V(K,t)?

Questions

2. What does V(K,t) measure?

Witten

$$\langle K \rangle = \int \mathrm{Tr}(\oint_K A)\ e^{2\pi i k\, CS(A)}\ DA$$

$$= (q^{1/2}+q^{-1/2})\ V(K, 1/q)$$

$$q = \exp(2\pi i/(2+k))$$

3. Can V(K,t) be found quickly by a quantum computer?

Braids

fig. 1 fig. 2 fig. 3

fig. 4 fig. 5 fig. 6
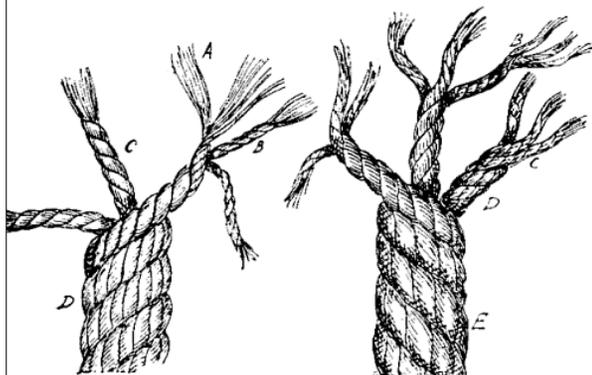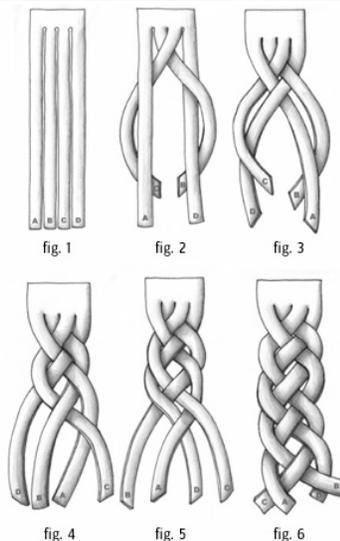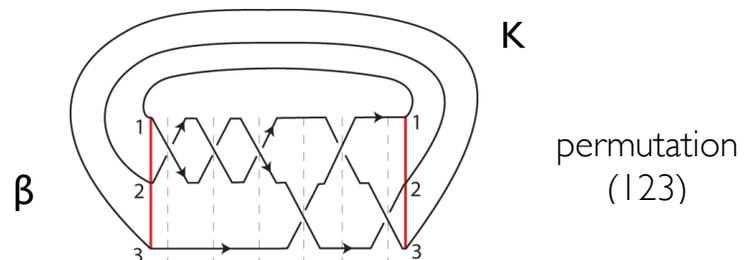
FIG. 1.—Construction of rope.

$P_n = \left\{ \begin{array}{c} \text{Polynomials p(x) of degree} \\ \text{n, with distinct roots} \end{array} \right\}$
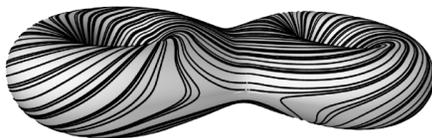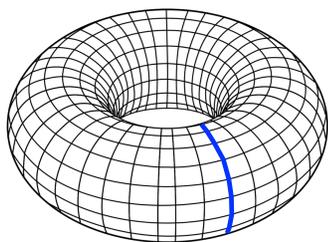
Braid group

$B_n = \pi_1(P_n)$

Braids as quantum permutations

K

β

permutation (123)

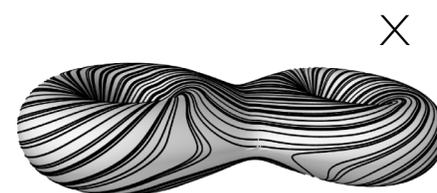$$V(K, t) = -(t^{1/2} + t^{-1/2})\, t^{\deg(\beta)}\, \mathrm{Tr}(\beta)$$

Fast computation for braid closures
… with few strands.

---

2. Knots on surfaces

Q. How to construct a random simple loop?

Hyperbolic Geometry

X

Closed loops, length < L: $\pi(L) \sim \dfrac{e^L}{L}$

Prime number theorem - Selberg *et al*

Simple loops, length < L: $\sigma(L) \sim C_X L^{6g-6}$
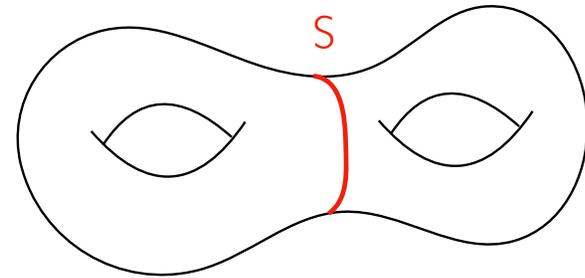
Mirzakhani 2008

Hyperbolic Geometry

X

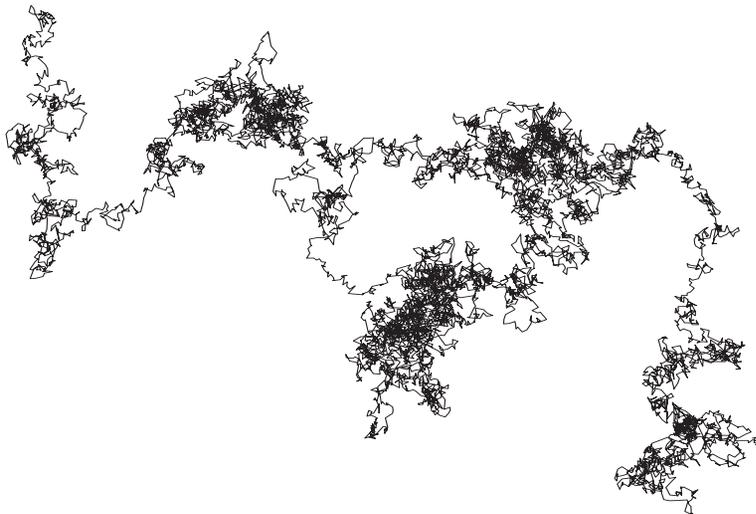A: Choose a long simple loop at random.

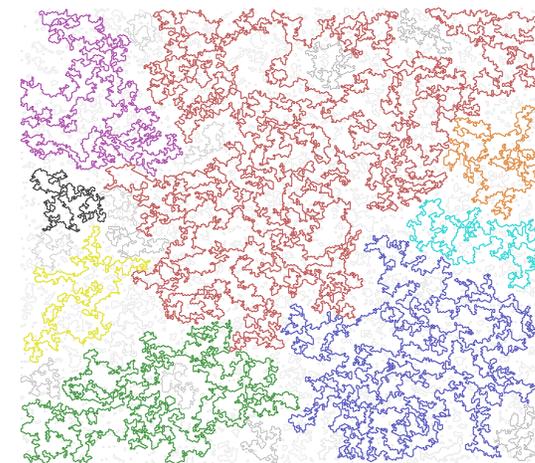Topological statistics universal

Answers are rational numbers

Example:

S

P(S cuts X into 2 pieces) = 1/49
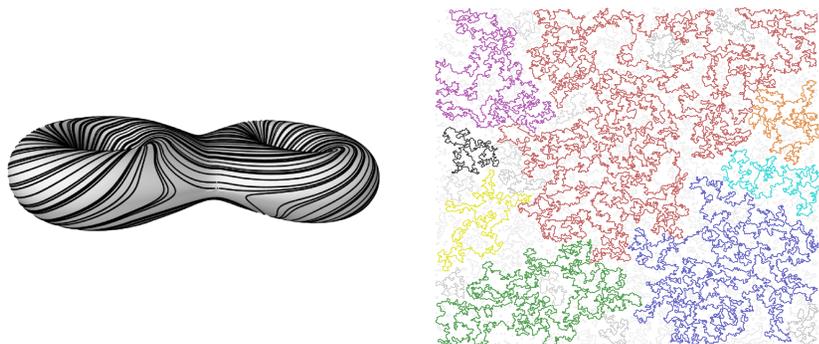
Use random paths?

Brownian motion (not simple)

Loop soups

Conformally invariant: level sets of GFF

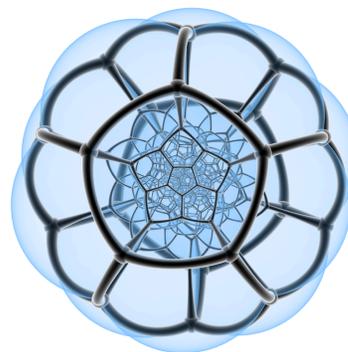Lawler, Werner, Schramm, Smirov, Sheffield, …

Q. Can these constructions be combined?

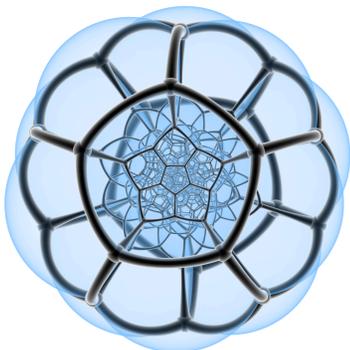3. Three-manifolds

Thurston, Perelman
1983-2006

All 3-manifolds are geometric.

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Thurston, Perelman
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3. Three-manifolds

Thurston, Perelman
1983-2006

Almost all 3-manifolds are hyperbolic.

Hyperbolic 3-manifolds

$$M^3 = \mathbb{H}^3/\Gamma$$

$$\Gamma \subset \mathrm{SL}_2(\mathbb{C})$$

Mostow rigidity
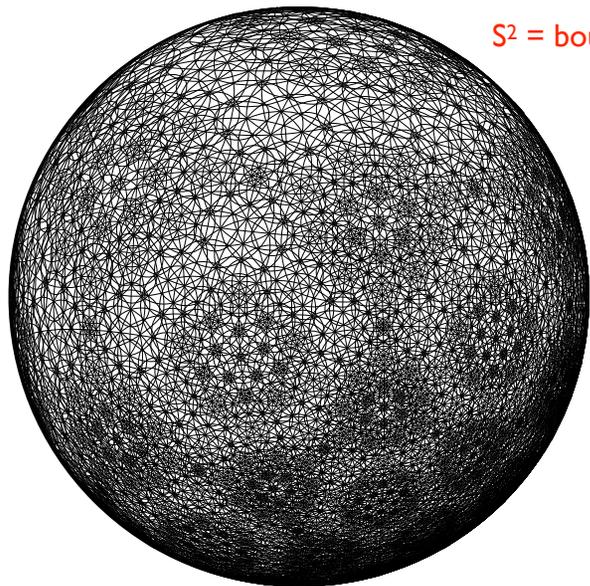
Arithmetic manifolds

Ratner's theorems

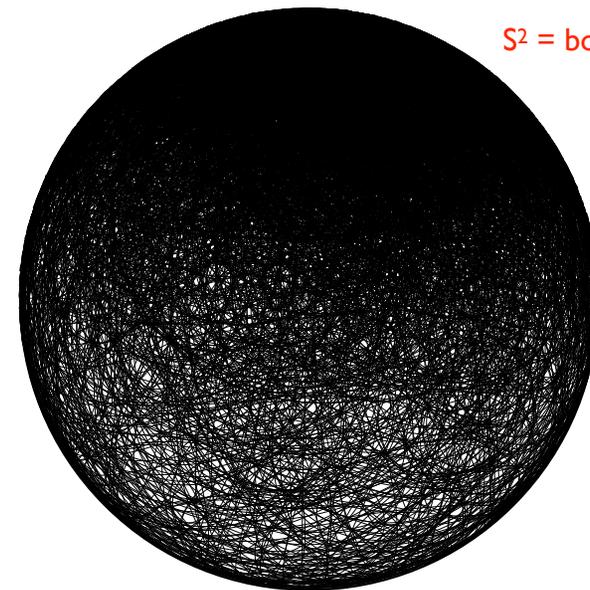## Planes in hyperbolic 3-manifolds

Given an isometric immersion

$$f \colon \mathbb{H}^2 \to \mathbb{H}^3/\Gamma$$

* It can happen that $f(\mathbb{H}^2)$ is a closed surface.
(miracle)

* Otherwise, $f(\mathbb{H}^2)$ is dense in M.

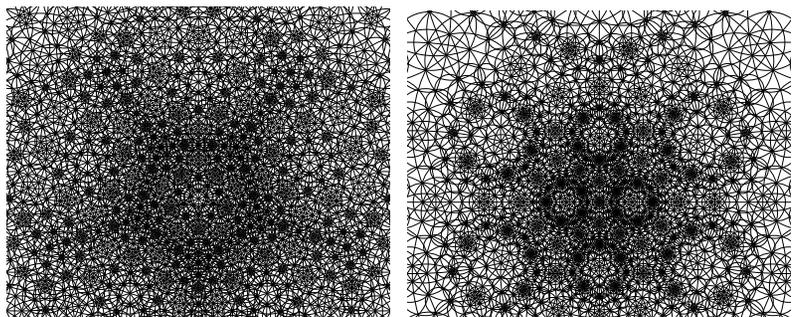*Shah, Margulis, Ratner 1991*

S² = boundary of $\mathbb{H}^3$

S² = boundary of $\mathbb{H}^3$

Open problem: Do ∞ many geodesic surfaces
⇒ M is arithmetic?

Yes! Margulis-Mohammadi, Fisher-Miller-Stover
2019

Mildly warped surfaces

Kahn—Markovic 2009

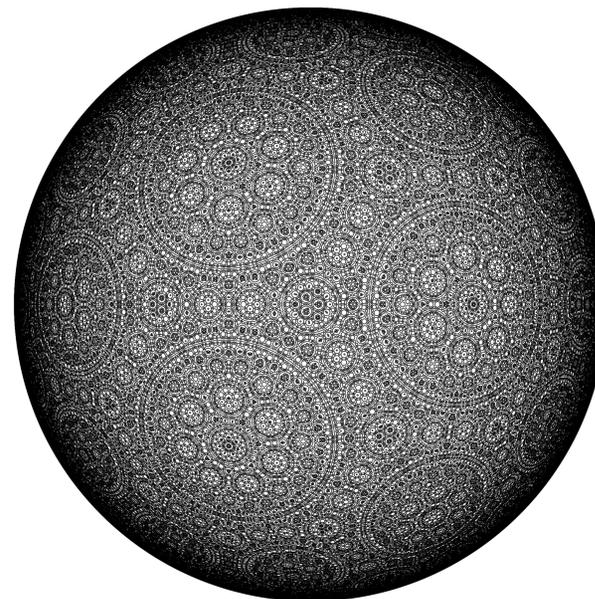Every M³ contains a *nearly* isometrically immersed surface S.

$$f : S = \mathbb{H}^2/\Delta \to M = \mathbb{H}^3/\Gamma$$

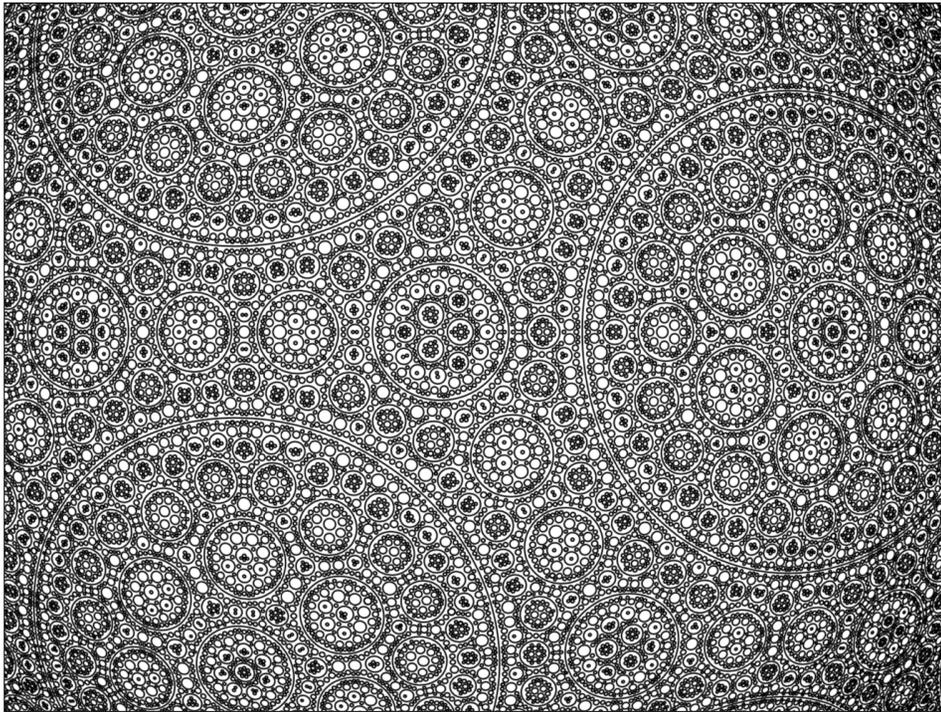S compact, but g(S) >> 0.

Topological finale

Agol—Wise 2012

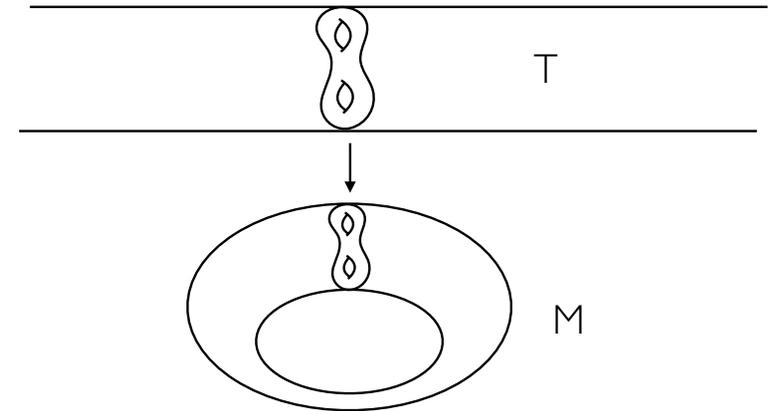After passing to a finite cover, M³ contains an *embedded* almost geodesic surface.

Topological finale, *bis*

M^3 has a finite cover that fibers over S^1.

This dubious-sounding question seems to have a definite chance for a positive answer.

Thurston, 1985

4. Planes in moduli spaces

For any *complex geodesic*

$$f : \mathbb{H}^2 \to \mathcal{M}_g,$$

the closure V of its image is an algebraic variety.

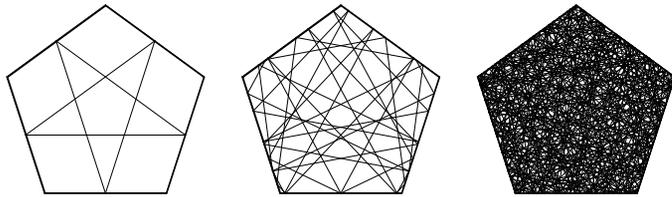*Eskin, Mirzakhani, Mohammadi; Filip 2016*

Q. What V are possible?

Teichmüller curves

There exist (rare jewels)

$$f : \mathbb{H}^2 \to \mathcal{M}_g,$$

such that the image V is an algebraic curve.

Veech, M, Bouw-Möller, ...

Teichmüller surfaces

There exist a complex geodesic

$$f : \mathbb{H}^2 \to \mathcal{M}_g$$

such that the image closure V
is a complex, totally geodesic surface.

M, Mukamel, Wright 2016

1879

A TREATISE

ON THE

HIGHER PLANE CURVES:

INTENDED AS A SEQUEL

TO

A TREATISE ON CONIC SECTIONS.

BY

GEORGE SALMON, D.D., D.C.L., LL.D., F.R.S.,

REGIUS PROFESSOR OF DIVINITY IN THE UNIVERSITY OF DUBLIN.

THIRD EDITION.

Dublin:
HODGES, FOSTER, AND FIGGIS, GRAFTON STREET,
BOOKSELLERS TO THE UNIVERSITY.
MDCCCLXXIX.

The solar configuration

Dawn

Sun

Codawn

V = codawn locus
in $\mathcal{M}_{1,3}$.
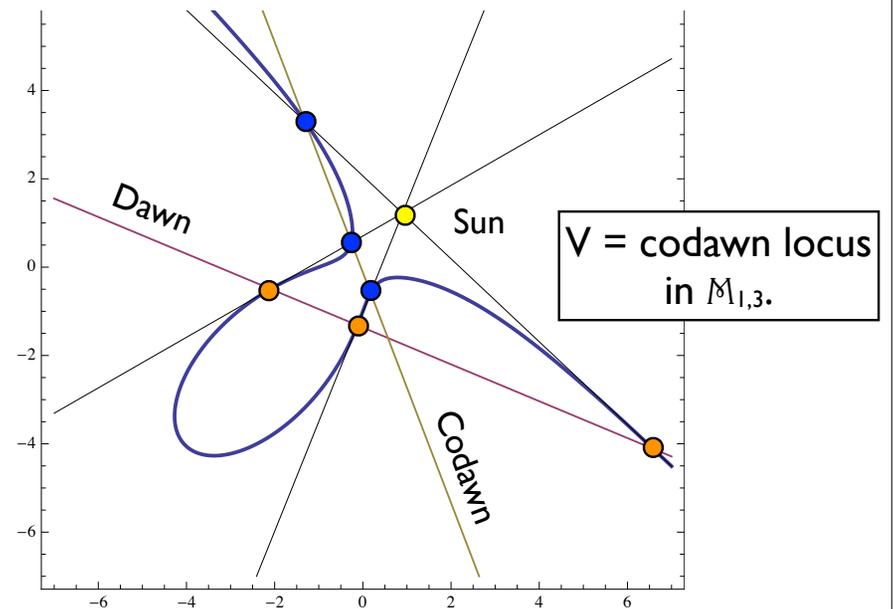
Flip side of rigidity.

Q. How to test if a real number x is in $\mathbb{Q}(\sqrt{D})$?

$$x = [a_0, a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

A. x is in $\mathbb{Q}(\sqrt{D})$ iff a_i's repeat.

Arithmetic chaos?

Does the number of $[a_1,\ldots,a_p]$ in $\in \mathbb{Q}(\sqrt{D})$ with $a_i \leq 2$ grow exponentially as the period p $\to \infty$?

Example:
[1], [1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 2, 1, 1, 1, 1, 2, 2], ...?...
lie in $Q(\sqrt{5})$.

Yes (mostly) for $\mathbb{Q}+\mathbb{Q}$; "Zaremba's conjecture",
Bourgain-Kontorovich 2011

5. Complex dynamics

Klein

Γ acting on $\widehat{\mathbb{C}}$

Quotient 3-manifold

Fatou and Julia

$f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$

?

Complex dynamics at MSRI

1985 (with moduli/string theory)

1995 (with Kleinian groups)

2022 (with stochastic complex analysis)

1986

1995

MATHEMATICAL SCIENCES
RESEARCH INSTITUTE

2022
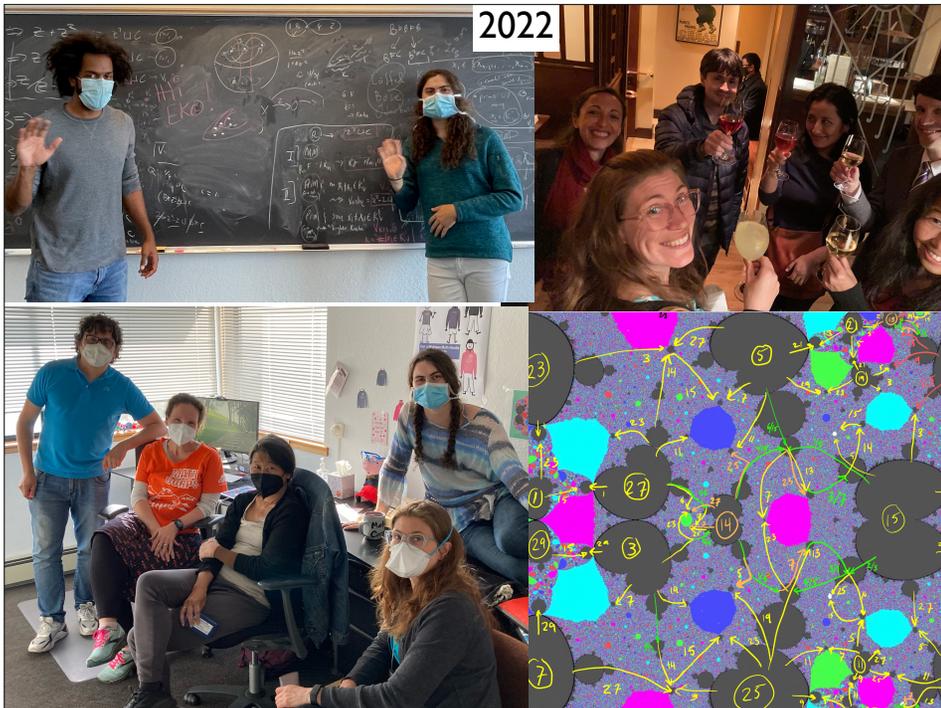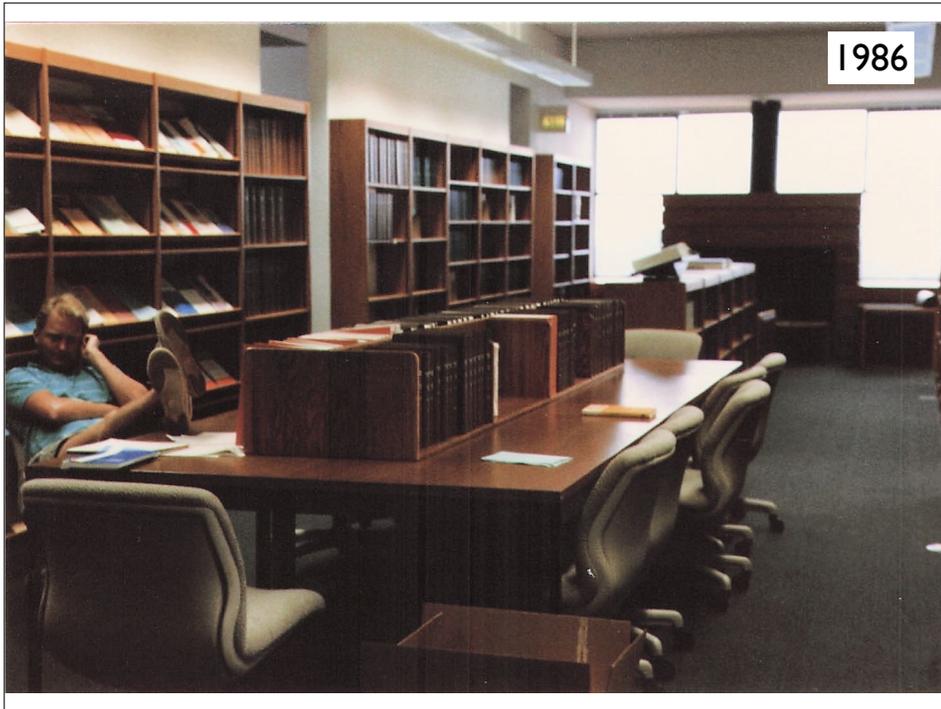
Dynamics of
a typical
rational map

$$f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

Attractor = 2 points

Dynamics of f(z) = z²
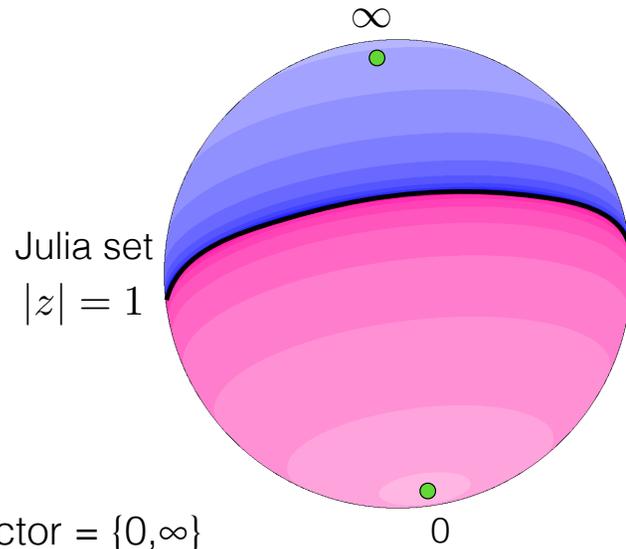
∞

Julia set
$|z| = 1$

Attractor = {0,∞}

0

The Mandelbrot set

$$f_c(z) = z^2 + c$$

$$0 \to c \to c^2 + c \to (c^2 + c)^2 + c \to \cdots$$

M = {c : orbit of 0 remains bounded}

The Mandelbrot set

$$f_c(z) = z^2 - c$$

$$0 \to -c \to c^2 - c \to (c^2 - c)^2 - c \to \cdots$$

M = {-c : orbit of 0 remains bounded}
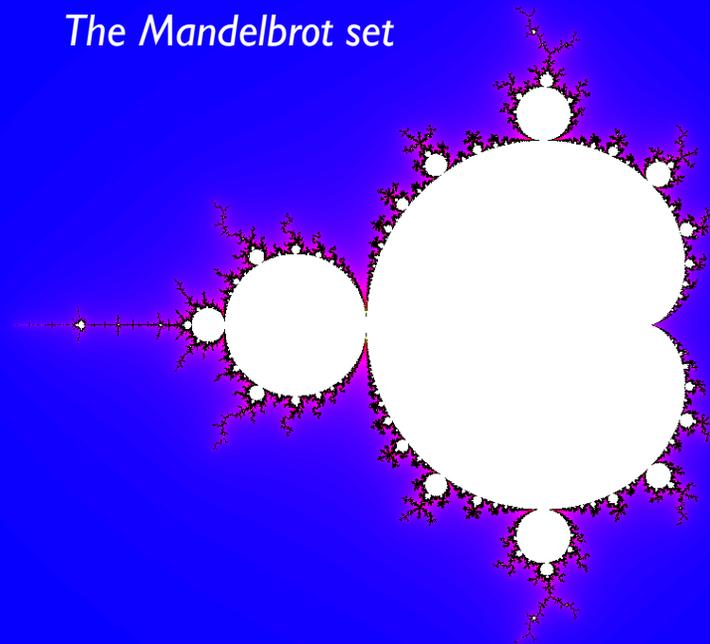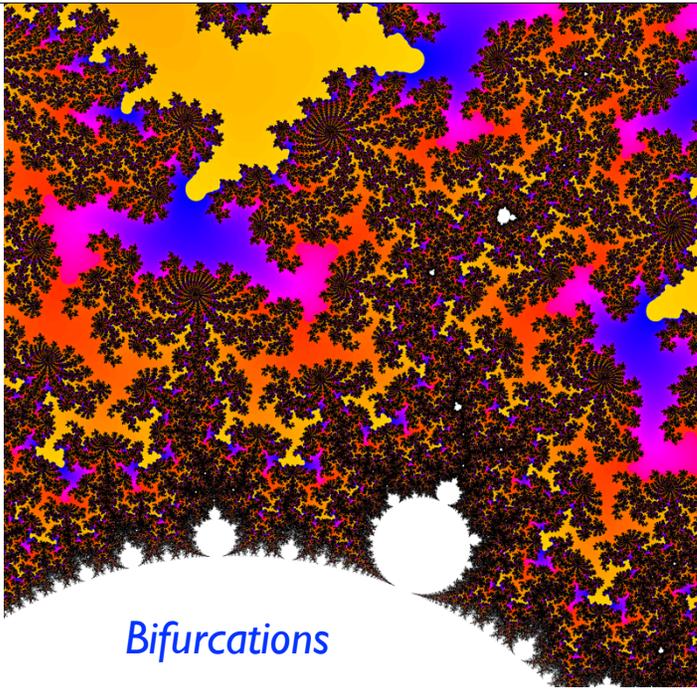
The Mandelbrot set

Bifurcations

Universality

A copy of the Mandelbrot set occurs near
any bifurcation
in any family of rational maps.

Dynamical Cyclotomy

$\Phi_n(z) = 0 \iff z$ has order n in S^1

$G_n(c) = 0 \iff 0$ has period n under $f_c(z) = z^2 + c$

$$\prod_{d|n} \Phi_d(z) = z^n - 1$$

$$\prod_{d|n} G_d(z) = f_c^n(0)$$

Main Conjecture
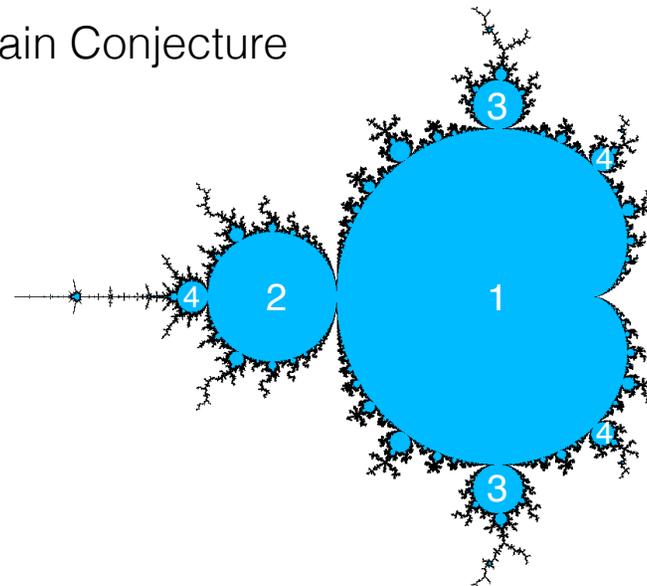
Every component of the interior of M
contains a zero of some $G_n(c)$.

Dynamical Cyclotomy

$G_1(c) = c$

$G_2(c) = c + 1$

$G_3(c) = c^3 + 2c^2 + c + 1$

$G_4(c) = c^6 + 3c^5 + 3c^4 + 3c^3 + 2c^2 + 1$

... ...

Q. Is $G_n(c)$ irreducible?

Dynamical Curves

Per_n = {degree 2 rational maps with one critical point of period n}

Q. (Milnor) Is Per_n irreducible?

Thm (Ramadas, 2022) If G_n is irreducible over $\mathbb{Q}$, then Per_n is irreducible over $\mathbb{C}$.

6. Solving polynomials

Problem: Find a root of

$x^d + a_1 x^{d-1} + a_2 x^{d-2} + \ldots + a_d = 0.$

What does `find' mean?

Which root?

Reliability?

Solving polynomials through the ages

Various authors

Solving the quadratic, circa 2000 BC
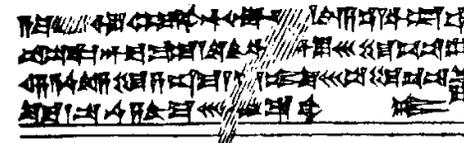
Solving the cubic, circa 1500 AD

Solving the quartic, circa 1500 AD

Insolvability of the quintic 1824

How to compute $\sqrt{2}$?

new guess for $\sqrt{2}$ = average of x + 2/x

1, 1.5, 1.41666, 1.414215, 1.4142136, ...

Newton's method $p \rightarrow N_p(z)$

Example of a *purely iterative algorithm.*

(Smale)

Newton's method for $x^2 - 2$
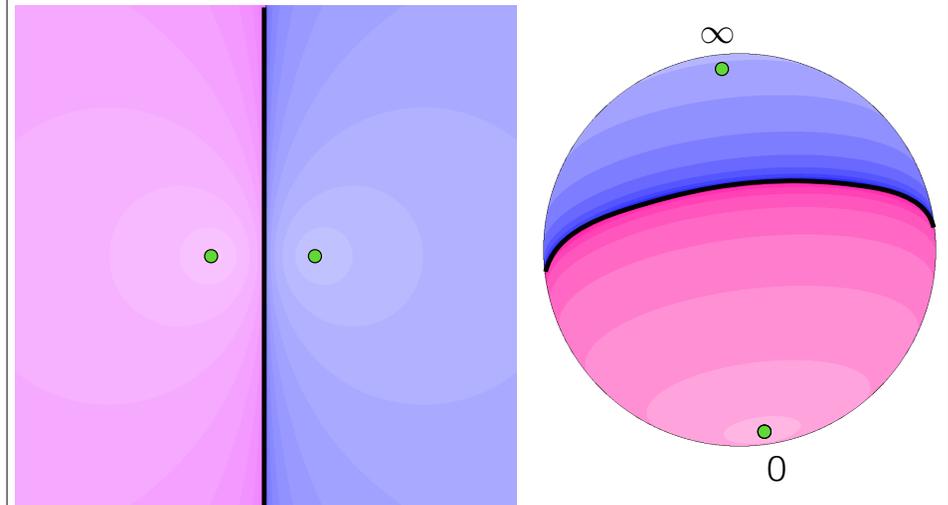

trivial family   *robust*

Cayley on cubics

The problem is to determine the regions of the plane such that P, taken at pleasure anywhere within one region, we arrive ultimately at the point A, anywhere within another region we arrive at the point B, and so for the several points representing the root of the equation. The solution is easy and elegant for the case of a quadric equation; but the next succeeding case of a cubic equation appears to present considerable difficulty.

from Sternberg, 2010

Newton's method for cubics

cubic Newton's method

not robust

The space of cubics

M is universal
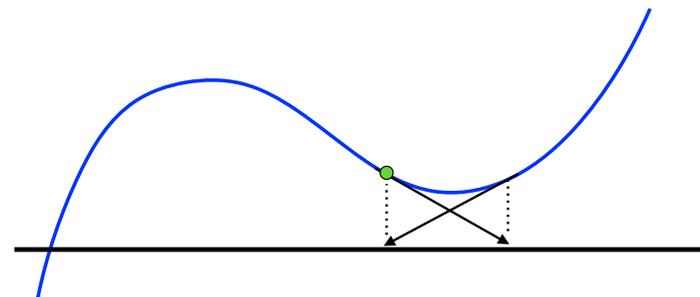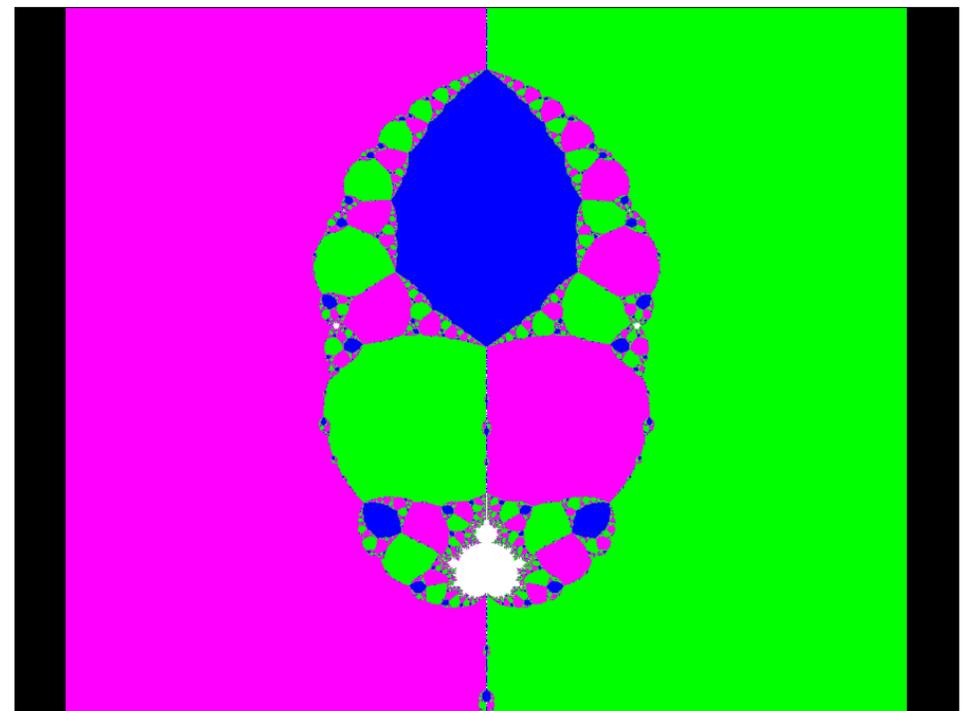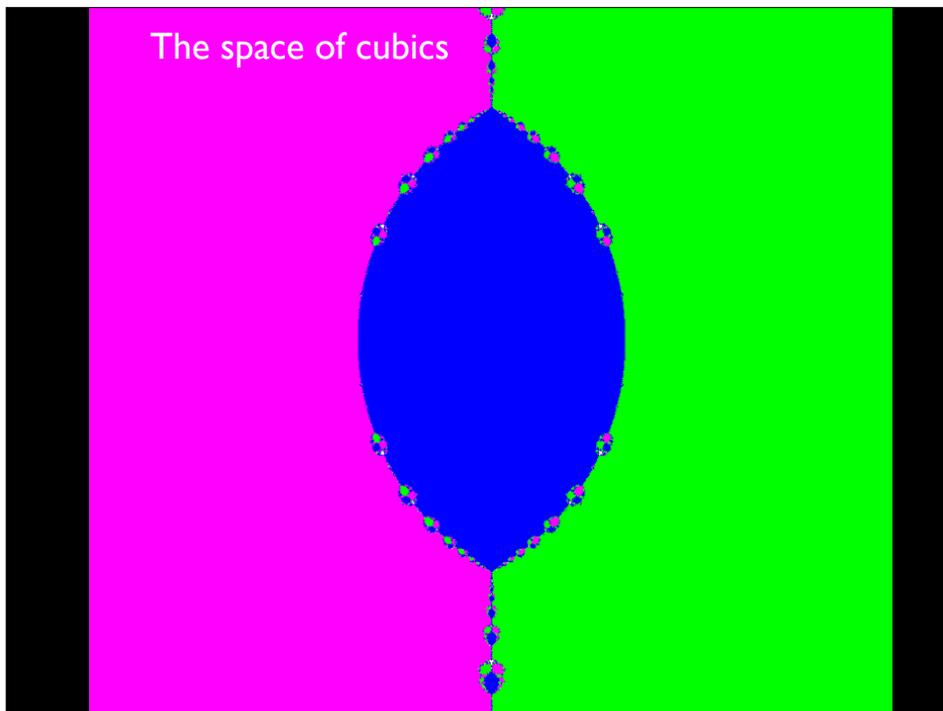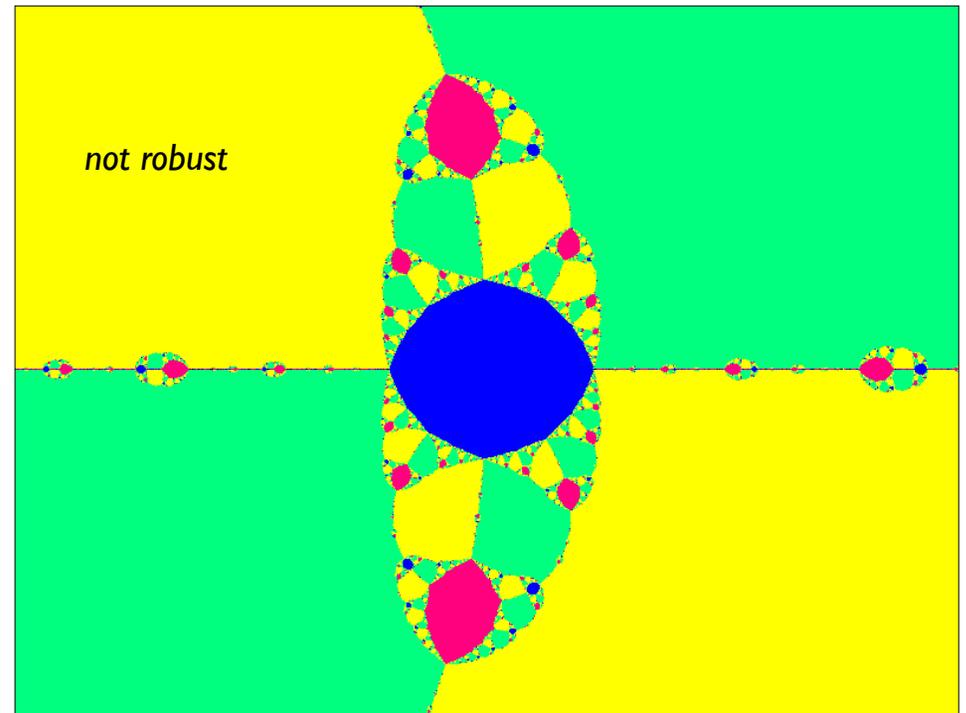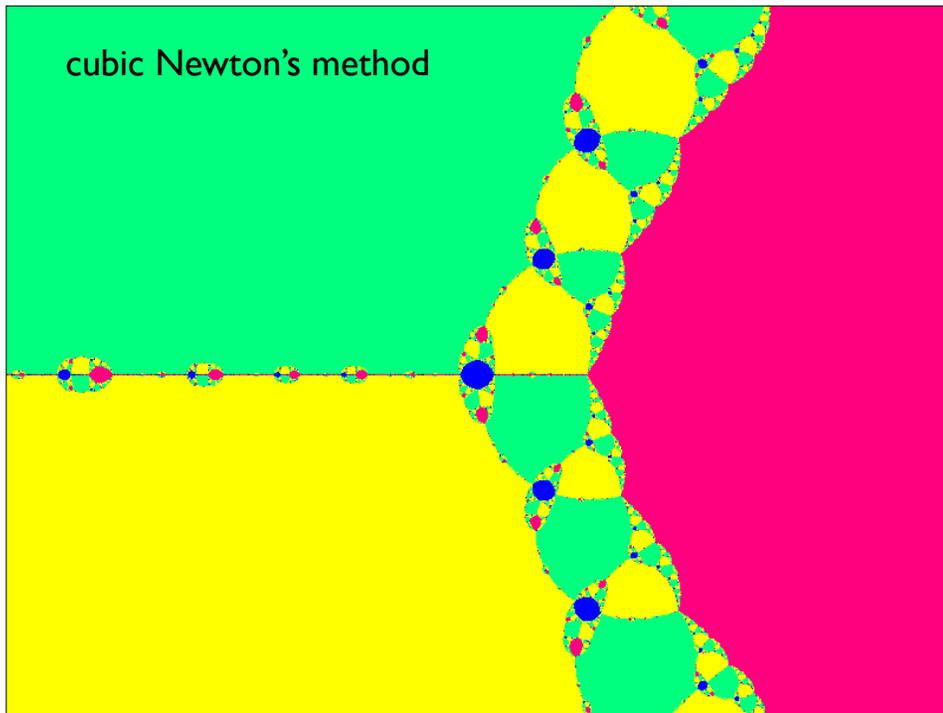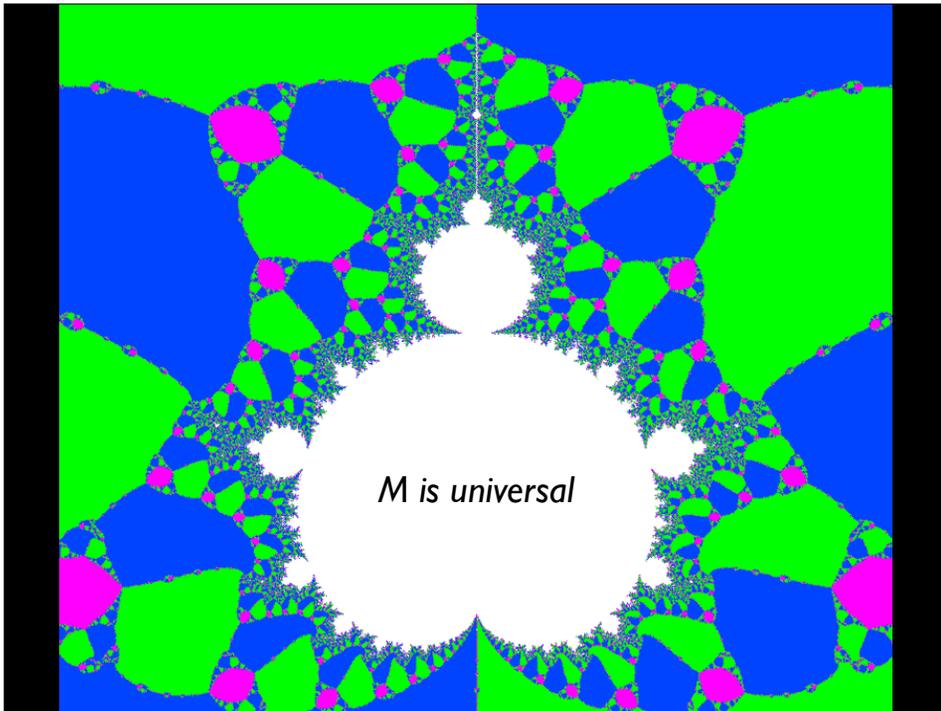
Unsolvability of the quartic

There is no robust, purely iterative algorithm to solve polynomials of degree 4 or more.

Location of failures

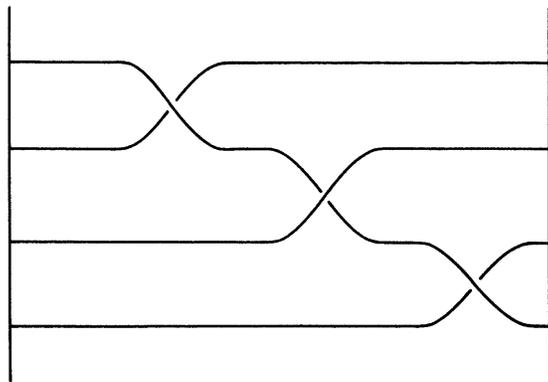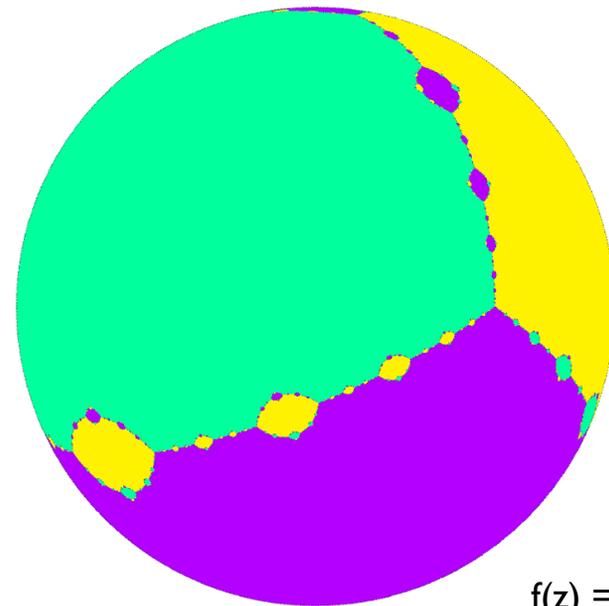Every algorithm fails somewhere along this loop in the space of degree 4 polynomials

Fig. 2.1. This braid does not arise for rational maps

What about cubics?

S₃ symmetry

$f(z) = (z^4 + 2z)/(2z^3+1)$

How to solve cubics

II. *There exists a unique degree 4 superconvergent algorithm for cubics. If the cubic polynomial p is given by*

$$p(X) = X^3 + aX + b$$

then the algorithm is given by

$$T_p(X) = X - \frac{(X^3 + aX + b)(3aX^2 + 9bX - a^2)}{(3aX^4 + 18bX^3 - 6a^2X^2 - 6abX - 9b^2 - a^3)}.$$

= *(Tate) Newton for*

$$q(X) = \frac{p(X)}{(3aX^2 + 9bx - a^2)}$$

Solvability of the quintic

The quintic can be solved by a tower of iterative algorithms.

But not the sextic, since there is no rational map with A_6 symmetry.

deg 11

Hilbert's 13th problem

(radicals $\sqrt[n]{z}$)

$\subset$ (iterative $\lim f^n(z)$)

$\subset$ (algebraic functions of 1 variable, $P^{-1}(z)$)

Q. Can sextic polynomials be solved using algebraic functions of 1 variable?

Arnold, Hironaka, Farb, Wolfsen, …

The future of
(low dimensional) mathematics

It involves computers.