

Appendix A: The Laws of Exponents

Exponents are a short-hand notation used to represent many factors multiplied together. All of the rules for manipulating exponents may be deduced from the laws of multiplication and division that you are already familiar with.

Exponential notation

Repeated multiplication is represented using exponential notation, for example:

$$3 \times 3 \times 3 \times 3 = 3^4.$$

There are four factors in the product, each of which is a 3. In the mathematical expression 3^4 , 4 is called the exponent and 3 is usually called the base.

For a real number a and a positive integer n , exponentials are defined by:

$$a^n = a \times a \times a \dots a \times a,$$

a total of n factors, each of which is an a .

A special case of exponential notation to note:

$$a^1 = a.$$

This makes sense, since a^1 should have only one factor of a .

Rules for combining exponents

Suppose that a and b are real numbers and that m and n are positive integers.

1. $a^n \cdot a^m = a^{n+m}.$

2. $\frac{a^n}{a^m} = a^{n-m}.$

3. $(a^n)^m = a^{n \cdot m}.$

4. $(a \cdot b)^n = a^n \cdot b^n.$

5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$

Justifications of these laws are based on the laws of multiplication and division that you are familiar with. Rules 1, 3 and 4 are justified below, and the ideas of the justifications for Rules 2 and 4 described. (The justifications of Rules 2 and 4 are exercises for this appendix.)

For the first exponential rule,

$$\begin{aligned}
a^n \cdot a^m &= (a \times a \times a \dots a \times a) \cdot (a \times a \times a \dots a \times a) \\
&\quad \text{total of } n \text{ factors} \quad \quad \text{total of } m \text{ factors} \\
&= a \times a \times a \dots a \times a \times a \times a \times a \dots a \times a \\
&\quad \text{total of } n + m \text{ factors} \\
&= a^{n+m}.
\end{aligned}$$

For the third exponential rule,

$$\begin{aligned}
(a^n)^m &= (a \times a \times a \dots a \times a) \cdot (a \times a \times a \dots a \times a) \cdot \dots \cdot (a \times a \times a \dots a \times a) \\
&\quad \text{total of } m \text{ sets of brackets} \quad \quad \text{each bracket has } n \text{ copies of } a \\
&= a \times a \times a \dots a \times a \cdot a \times a \times a \dots a \times a \cdot \dots \cdot a \times a \times a \dots a \times a \\
&\quad \text{the total number of copies of } a \text{ is } m \cdot n \\
&= a^{mn}.
\end{aligned}$$

For the fourth exponential rule,

$$\begin{aligned}
(a \cdot b)^n &= (a \cdot b) \times (a \cdot b) \times (a \cdot b) \times \dots \times (a \cdot b) \\
&\quad n \text{ copies of } (a \cdot b) \text{ so there are } n \text{ copies of } a \text{ and } n \text{ copies of } b \\
&= (a \times a \times a \times \dots \times a) \cdot (b \times b \times b \times \dots \times b) \\
&\quad n \text{ copies of } a \quad \quad n \text{ copies of } b \\
&= a^n \cdot b^n.
\end{aligned}$$

The idea for justifying second exponential rule is this: there are n copies of a in the numerator of the fraction, and m copies of a in the denominator of the fraction. Some of these copies of a will cancel out.

Lastly, the idea for the justification of the fifth exponential rule is this: there are n copies of the fraction $\frac{a}{b}$. Thus, the numerator will consist of n copies of a multiplied together, so the numerator is a^n . The denominator will consist of n copies of b multiplied together, so the denominator is b^n . (To review concepts such as numerator, denominator and multiplication of fractions, see Appendix E.)

Many of these rules also work when m and n are not positive integers - for example, if m and n are fractions. Some special considerations apply when m and n are not positive integers in order to make sure that the expression makes good mathematical sense. For example, while it is possible to write down a collection of symbols like:

$$(-2)^{1/2} = \sqrt{-2}.$$

The collection of symbols $(-2)^{1/2}$ should represent the number that is the square root of negative two. However, since squaring any real number gives a result that is greater than or equal to zero, there is no real number that, when squared, gives -2 . While it is possible to write down the collection of symbols: $(-2)^{1/2}$, there is no real number that equals this collection of symbols.

Example A.1

Use the laws of exponents to simplify the following algebraic expressions when possible.

a) $4b \cdot (3b)^5$.

b) $(2x^3)^2 \cdot \left(\frac{3}{x^2}\right)^4$.

Solution:

a) $4b \cdot (3b)^5 = 4b \cdot 3^5 \cdot b^5 = 4 \cdot 243 \cdot b \cdot b^5 = 972 \cdot b^6$.

b) $(2x^3)^2 \cdot \left(\frac{3}{x^2}\right)^4 = 2^2 \cdot (x^3)^2 \cdot \left(\frac{3^4}{(x^2)^4}\right) = 4 \cdot 81 \cdot x^6 \cdot \frac{1}{x^8} = 324 \cdot x^{6-8} = 324 \cdot x^{-2}$.

Part (b) illustrates an interesting point - the application of the exponent rules can generate algebraic expressions such as x^{-2} , which we have no way of interpreting at this point.

Interpreting and simplifying fractional and negative exponents

The way that we defined exponential notation as a short-hand for repeated multiplication makes complete sense for exponents that are positive integers. However, exactly what an expression like x^{-2} should mean is not so clear.

Let a be a real number, and let m and n be positive integers.

$$a^0 = 1 \quad (\text{provided that } a \neq 0)$$

$$a^{-n} = \frac{1}{a^n} \quad (\text{provided that } a \neq 0)$$

$$a^{1/n} = \sqrt[n]{a} \quad (\text{if } n \text{ is an even integer, then this only makes sense when } a \geq 0)$$

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \quad (\text{if } n \text{ is an even integer, then this only makes sense when } a \geq 0)$$

Example A.2

Using the laws of exponents, simplify the following expressions as much as possible.

$$\text{a) } \left(\frac{3^{-1} \cdot L}{L^{-2}} \right)^{-3}.$$

$$\text{b) } (2w^2y^3)^{-1} \cdot \frac{1}{y^{-8}}.$$

$$\text{c) } (a^2 + r^2)^2.$$

Solution:

$$\text{a) } \left(\frac{3^{-1} \cdot L}{L^{-2}} \right)^{-3} = \frac{(3^{-1} \cdot L)^{-3}}{(L^{-2})^{-3}} = \frac{(3^{-1})^{-3} \cdot L^{-3}}{L^6} = 3^3 \cdot L^{-3-6} = 27 \cdot L^{-9}.$$

b)

$$(2w^2y^3)^{-1} \cdot \frac{1}{y^{-8}} = 2^{-1} \cdot (w^2)^{-1} \cdot (y^3)^{-1} \cdot \frac{1}{y^{-8}} = \frac{1}{2} \cdot w^{-2} \cdot y^{-3} \cdot \frac{1}{y^{-8}} = \frac{1}{2} \cdot w^{-2} \cdot y^{-3-(-8)} = \frac{1}{2} \cdot w^{-2} \cdot y^5$$

$$\text{c) } (a^2 + r^2)^2 = (a^2 + r^2) \cdot (a^2 + r^2) = a^4 + 2a^2r^2 + r^4.$$

The important point in Part (c) is that there is no exponential rule that is valid when the terms inside the bracket are added (rather than multiplied or divided) together. In particular,

$$(a + b)^n \neq a^n + b^n.$$

The important thing to remember here is that $(a + b)^n$ and $a^n + b^n$ are not the same. (To review the expansion of complicated algebraic expressions, see Appendix B.)

Example A.3

$$\text{a) } \sqrt[3]{8w^{18}}.$$

$$\text{b) } \sqrt{\frac{64k^2}{9T^9}}.$$

$$\text{c) } \sqrt[n]{a^n}.$$

Solution:

$$\text{a) } \sqrt[3]{8w^{18}} = \sqrt[3]{8} \cdot \sqrt[3]{w^{18}} = 2 \cdot w^6.$$

$$\text{b) } \sqrt{\frac{64k^2}{9T^9}} = \frac{\sqrt{64k^2}}{\sqrt{9T^9}} = \frac{\sqrt{64} \cdot \sqrt{k^2}}{\sqrt{9} \cdot \sqrt{T^9}} = \frac{8k}{3T^{9/2}}.$$

c) The simplification of $\sqrt[n]{a^n}$ is not completely straight-forward. If a is a positive number or zero, then

$$\sqrt[n]{a^n} = (a^n)^{1/n} = a^1 = a$$

as you might expect. If a is a negative number, then the situation is more complicated. If n is an odd number, then $\sqrt[n]{a^n} = a$. To illustrate this consider the case where $n = 3$ and $a = -2$,

$$\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2.$$

If n is an even number, then $\sqrt[n]{a^n} = |a|$. To illustrate this consider the case where $n = 2$ and $a = -2$,

$$\sqrt{(-2)^2} = \sqrt{4} = 2.$$

Exercises for Appendix A

For Problems 1-10, evaluate the quantity (if possible) without using a calculator.

1. 7^2 .
2. 3^0 .
3. $\frac{9^2}{9^3}$.
4. $\sqrt{9^9}$.
5. $(-1)^4$.
6. $(-1)^5$.
7. $\frac{(12)^3}{(-12)^3}$.
8. $\sqrt[3]{8}$.
9. $\sqrt[3]{-8}$.
10. $\sqrt{(-8)^2}$.

For Problems 11-20, simplify the expression as much as possible.

11. $e \cdot e^{2t} \cdot e^t \cdot 2^e$.

$$12. \frac{x^{2n+1} \cdot (2y)^{n+1}}{(x^2 \cdot y)^n}.$$

$$13. (a-b)^{5/2} \cdot \sqrt{a+b}.$$

$$14. (T^2 \cdot w^4)^{1/2}.$$

$$15. e \cdot \sqrt[3]{e^{3r}}.$$

$$16. \left(\frac{2w^2 \cdot y^{3/2}}{\sqrt{w+y}} \right)^2.$$

$$17. (C \cdot e^{wt})^2.$$

$$18. (5xy)^{-1} \cdot (xy^2)^2.$$

$$19. \sqrt{36u^2v}.$$

$$20. \frac{8P^{-2}}{4P^{-4}}.$$

For Problems 21-25, decide whether each of the statements are true or false.

$$21. (a+b)^2 = a^2 + b^2.$$

$$22. 10q^{-2} = \frac{1}{10q^2}.$$

$$23. \frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}.$$

$$24. x^n \cdot x^m = x^{n+m}.$$

$$25. z^2 + z^2 = z^4.$$

In problems 26 and 27 you will justify the last two exponential rules.

26. Suppose that a is a real number, and that n and m are positive integers. Explain why:

$$\frac{a^n}{a^m} = a^{n-m}.$$

27. Suppose that a and b are real numbers, and that n is a positive integer. Explain why:

$$\left(\frac{a}{b} \right)^n = \frac{a^n}{b^n}.$$

Answers to Exercises for Appendix A

1. 49.
2. 1.
3. $1/9$.
4. 81.
5. 1.
6. -1.
7. -1.
8. 2.
9. -2.
10. 8.
11. $2^e \cdot e^{3t+1}$.
12. $2^n xy$.
13. $(a + b)^3$.
14. $|T| \cdot w^2$.
15. e^{r+1} .
16. $(4w^4y^3)/(w + y)$.
17. $C^2 \cdot e^{2wt}$.
18. $(1/5)xy^3$.
19. $6 \cdot |u| \cdot (v)^{1/2}$.
20. $2P^2$.
21. False.
22. False.
23. False.
24. False.
25. False.
26. The basic idea here is cancellation of common factors from the numerator and denominator. There are n factors of a in the numerator and m factors of a in the denominator. In the case where $n > m$, all of the factors of a in the denominator will cancel with factors of a in the numerator, leaving $n - m$ factors of a remaining in the numerator. In the case where $n < m$, the n factors of a in the numerator will cancel with factors in the denominator, leaving $m - n$ factors of a in the denominator. Thus, the simplified expressions will be: $(1/a)^{m-n} = a^{-1(m-n)} = a^{n-m}$.

27. The basic idea here is that you will have n factors, each of which is a/b . Thus, the overall numerator will consist of n copies of a all multiplied together, i.e. the overall numerator is a^n . Similarly, the overall denominator is b^n .