Name: Solutions

Instructor: ____________

MATHEMATICS Xa

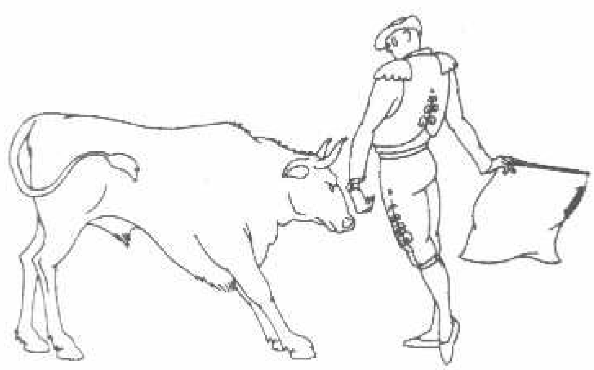
Final Examination

January 24, 1996

Please show all your work on this exam paper. You must explain your reasoning fully and clearly in order to get full credit. If you do any work on the back of a page, write us a note on the exam cover letting us know.

Work carefully, think clearly, and DO WELL!!

Question	Points	Score
1	10	
2	11	
3	6	
4	7	
5	13	
6	7	
7	13	
8	8	
9	9	
10	6	
11	10	
Total	100	



Use your time wisely!

Barták / Nebelspalter Zurich

1. (10 points) Differentiate the following: (You need not simplify your answers.)

a) $f(x) = x^3 + 3^x + e^{\pi}$

$3x^2 + \ln 3 \cdot 3^x$

b) $f(x) = 5(x^2 + 3x + 17)^{10}$

$50(x^2+3x+17)^9(2x+3)$

c) $f(x) = 2x\, e^{x^3}$

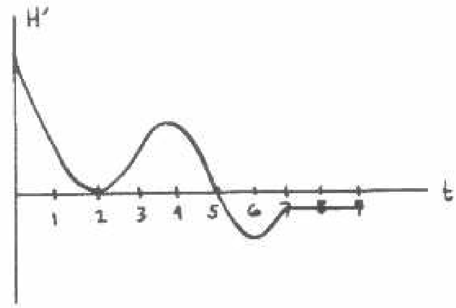
$2x \cdot e^{x^3} \cdot 3x^2 + 2e^{x^3}$

d) $f(x) = \dfrac{\ln(x^{-1})}{e^{3x}}$

$\left[\ln\left(\frac{1}{x}\right)\right]' = \frac{1}{1/x} \cdot -x^{-2} = \frac{-x}{x^2} = -\frac{1}{x}$

$$\frac{-e^{3x} \cdot \frac{1}{x} - \ln\left(\frac{1}{x}\right) \cdot e^{3x} \cdot 3}{e^{6x}}$$

2. (11 points) Let H(t) be the height of water in a reservoir at time t. Below is the graph of H '(t) (NOT H(t)).



Circle the appropriate answer:

a) At which value of t is the height of water in the reservoir the greatest? 0 1 2 3 4 5 6 7 8 9

b) At which value of t is the height of the water in the reservoir ~~the least~~ decreasing most rapidly? 0 1 2 3 4 5 6 7 8 9

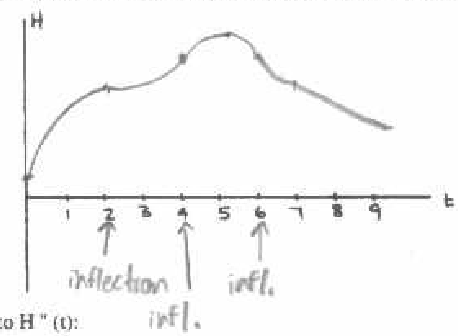
The following questions refer to the graph of H(t), the height of the water in the reservoir at time t and should be answered considering the time interval $0 < t < 9$.

c) Where is the graph of H increasing? [0, 5]

d) Where is the graph of H concave up? [2, 4], [6, 7]

e) Where is the graph of H concave ***down*** and increasing? [0, 2], [4, 5]

f) Sketch a rough graph of H(t) assuming that at t=0 H > 0. (Your graph should be consistent with your answers to the previous questions.) Please label the t coordinates of the points of inflection.



The following questions refer to H " (t):

g) Where is H "(t) zero ?

t = 2, 4, 6

h) Where is H "(t) negative?

[0, 2], [4, 6]

3. (6 points) Find the following limits:

a) $\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} - f'(x)$

$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

So the limit is $\boxed{0}$.

b) $\lim_{x \to \infty} \frac{3x^2 + 2x - 6}{2x^2}$

$$= \lim_{x \to \infty} \left(\frac{3x^2}{2x^2} + \frac{2x}{2x^2} - \frac{6}{2x^2} \right) = \lim_{x \to \infty} \left(\frac{3}{2} + \frac{1}{x} - \frac{3}{x^2} \right) = \boxed{\frac{3}{2}}$$

(or use l'Hôpital's rule twice)

c) $\lim_{h \to 0} \frac{e^{2+h} - e^2}{h}$

$= f'(2)$ where $f(x) = e^x$

$f'(e^x) = e^x$, so $\lim = f'(2) = \boxed{e^2}$

4. (7 points) a) Give the limit definition of f '(2).

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

b) Use your answer to (a) to find f '(2) for $f(x) = \frac{3}{2+5x}$. You must show all your algebra clearly and explicitly. The quality of the write-up is important. After having completed the problem, determine whether or not your answer is correct.

$$f'(2) = \lim_{h \to 0} \frac{\frac{3}{2+5(x+h)} - \frac{3}{2+5x}}{h} = \lim \frac{3(2+5x) - 3(2+5x+5h)}{h(2+5x+5h)(2+5x)}$$

$$= \lim \frac{6+15x-6-15x-15h}{h(2+5x+5h)(2+5x)} = \lim \frac{-15}{(2+5x+5h)(2+5X)} =$$

$$= \frac{-15}{(2+5x)^2} \underset{x=2}{\uparrow}{=} \frac{-15}{144}$$

Using chain rule: $f(x) = 3(2+5x)^{-1}$

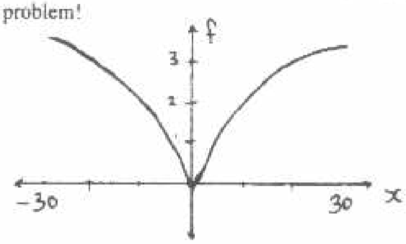
$$f'(x) = -3(2+5x)^{-2} \cdot 5 = \frac{-15}{(2+5x)^2}$$

$$\text{so } f'(2) = \frac{-15}{144}$$

and the answer is correct.

5. (13 points) f is a continuous function of x. Below is a table of values for f (rounded off) and a graph of f. Be sure to show all your work in this problem!

x	f(x)
9.8	2.7876
9.9	2.2976
10.0	2.3076
10.1	2.3174
10.2	2.3272
10.3	2.3368



a) Using the table above, find upper and lower bounds for the instantaneous rate of change of f at x = 10. Your answers should differ from one another by no more than .005. (Show your work!)

slope decreasing, so $f'(10) \le \dfrac{2.3076 - 2.2976}{10.0 - 9.9} = .1$

$f'(10) \ge \dfrac{2.3174 - 2.3076}{10.1 - 10.0} = .098$

Answer: $.098 < f'(10) < .1$

b) Actually, f is given explicitly by $f(x) = \ln\sqrt{x^2+1}$. Find the exact value (not a decimal approximation) of $f'(10)$. Make sure your answer to b) is consistent with your answer to a).

$$f'(x) = \frac{1}{\sqrt{x^2+1}} \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x}{x^2+1} \underset{x=10}{=} \frac{10}{101} \quad (\approx .099)$$

c) Find the equation of the line tangent to f(x) at x = 10.

slope $= \dfrac{10}{101}$ point $= (10, 2.3076)$. So use point-slope formula: $(y - 2.3076) = \dfrac{10}{101}(x - 10)$

d) The resolution of the picture given in part (a) is not good enough to determine whether or not $f'(0)$ is defined. Is it defined? If so, what is its value?

$f'(x) = \dfrac{x}{(x^2+1)}$, so $f'(0) = \dfrac{0}{1} = 0$

The derivative is defined at 0.

e) We compute $f''(x) = \dfrac{-x^2+1}{(x^2+1)^2}$. Find the **x** ***and*** **y** coordinates of all inflection points of f. Explain how you know that these points are actually points of inflection (what is your criterion?).

inflection points are where $f''(x) = 0$

$\dfrac{-x^2+1}{(x^2+1)^2} = 0$ when $-x^2+1 = 0 \Rightarrow x = \pm 1$

$x=1$: when $x = .999$, $f''(x) > 0$. When $x = 1.001$, $f''(x) > 0$ so is an inflection point. $y = \ln\sqrt{2}$.

$x=-1$: when $x = -1.001$, $f''(x) < 0$. when $x = -.999$, $f''(x) > 0$. So is an inflection point. $y = \ln\sqrt{2}$

6. (7 points) Match to each equation a possible graph. A and B are positive constants. A >1, B > 1.

a) $y = \dfrac{2}{(x+A)(x-B)}$ vi

b) $y = \dfrac{x^2}{(x+A)(x-B)}$ v

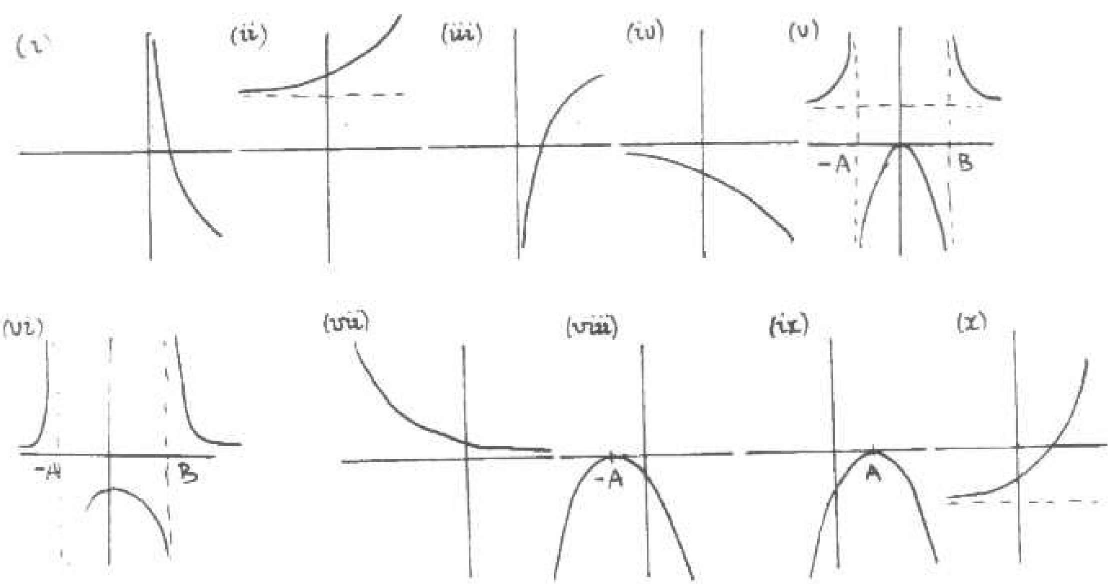
c) $y = -(x-A)^2$ ix

d) $y = A^x + B$ ii

e) $y = -B\ln x$ i

f) $y = y = e^{-Ax}$ vii

g) $y = -A e^x$ iv



7. (13 points) $f(x) = \frac{\ln x}{x^2}$

a) Find the roots of f(x).

$\frac{\ln x}{x^2} = 0$ when $\ln x = 0$, so $x = 1$

b) Find ALL critical points of f on the interval $[e^{0.1}, e^{10}]$. Give exact values, *not* approximations from your calculator.

Where is the absolute maximum of the function on the interval $[e^{0.1}, e^{10}]$.

Where is the absolute minimum of the function on the interval $[e^{0.1}, e^{10}]$.

Please explain your reasoning clearly and completely.

$f'(x) = \frac{1}{x} \cdot \frac{1}{x^2} + \ln x \cdot \left(\frac{-2}{x^3}\right) = \frac{1}{x^3} - \frac{2\ln x}{x^3}$. So set $\frac{1}{x^3} - \frac{2\ln x}{x^3} = 0 \Rightarrow$

$2\ln x = 1 \Rightarrow \ln x = \frac{1}{2} \Rightarrow \boxed{x = e^{1/2}} \to$ critical point

$f(e^{1/2}) = .184$ check endpoints: $f(e^{.1}) = .082$

$f(e^{10}) = 2.06 \times E^{-8}$

$\left.\begin{array}{l} f'(e^{1/2} - h) > 0 \\ f'(e^{1/2} + h) < 0 \end{array}\right\}$ so is a max

So $e^{1/2}$ gives absolute max and e^{10} gives absolute min over interval

critical points at x = $e^{1/2}$

absolute maximum at x = $e^{1/2}$

absolute minimum at x = e^{10}

8. (8 points) Craig deposits $410 Canadian dollars into an account with an interest rate of 4% per year compounded annually. On the same day, Rob deposits 400 Canadian dollars into an account with interest rate 4% per year compounded quarterly. When will Rob and Craig have the same amount of money in their accounts? (Assume they make no further deposits or withdrawals.) Please show all your work. Please give an exact answer and then give a decimal approximation.

$C = 410 \cdot (1.04)^t$ need to solve

$R = 400 \cdot (1.01)^{4t}$ $\Rightarrow$ $410 \cdot (1.04)^t = 400 \cdot (1.01)^{4t}$

$$\ln 410 + t \ln(1.04) = \ln 400 + 4t \ln(1.01)$$

$$\ln 410 - \ln 400 = t(4 \ln 1.01 - \ln 1.04) \Rightarrow t = \frac{\ln 410 - \ln 400}{4 \ln 1.01 - \ln 1.04} = 42.53 \text{ yrs}$$

Their balances will be equal ____________ years after the original date of deposit.

Decimal approximation: ____________ years

9. (9 points) A university baseball stadium has 4000 seats, all equally priced. At $6.00 per seat, only 2100 seats are taken. For every $1 drop in price, the university figures that 700 more people will attend the game. By how much should the price of the ticket be reduced in order to maximize the ticket revenue for the game? **Explain your reasoning clearly and completely.** $x = \$\text{reduction}$

$$\text{Revenue} = (2100 + 700x)(6 - x) = -700x^2 + 2100x + 12600$$

$$R'(x) = -1400x + 2100 = 0 \Rightarrow x = 1.5$$

This is a max b/c $R'(1.4) > 0$, $R'(1.6) < 0$.

So reduce ticket price by $1.50

10. (6 points) Solve for x:

a) $e^{x^2 + 2x} = 4$

quadratic eq.

$$x^2 + 2x = \ln 4 \Rightarrow x^2 + 2x - \ln 4 = 0 \Rightarrow x = \frac{-2 \pm \sqrt{4 + 4\ln 4}}{2}$$

so $x = .55$ or $x = -2.55$

b) $\ln(x+5) - \ln(2x+1) = 3$

$$e^{\ln(x+5) - \ln(2x+1)} = e^3 \Rightarrow e^{\ln(x+5)} \cdot e^{-\ln(2x+1)} = e^3$$

so $$\frac{(x+5)}{(2x+1)} = e^3 \Rightarrow x + 5 = 2e^3 x + e^3 \Rightarrow (1 - 2e^3)x = e^3 - 5$$

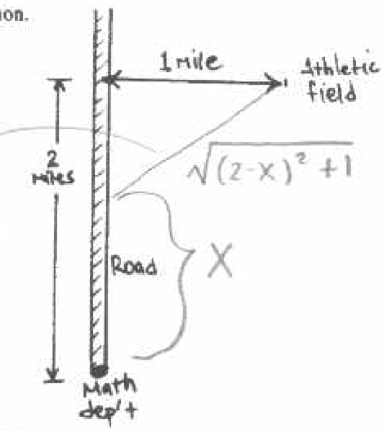
$$x = \frac{e^3 - 5}{(1 - 2e^3)}$$

11. (10 points) Due to popular demand, a bicycle path is to be constructed connecting the mathematics department to the athletic center. The mathematics department is located on a straight road, as drawn. The athletic center is in a field 1 mile from the road. The bike path will follow the road for a while and then will cut off into the field and make a beeline for the athletic center. Constructing the path costs K \$/mile along the road and 2K \$/mile across the field.

a)Express the cost of constructing the path as a function of x, the distance the path follows alongside the road. Your answer will involve the positive constant K. Please specify the domain of the cost function.

$C = Kx + 2K\sqrt{x^2-4x+5}$

(Pythagorus)



Answer: C(x) = $Kx+2K\sqrt{x^2-4x+5}$ domain: $0 \le x \le 2$

b) Suppose that K = 400. With the aid of your graphing calculator, approximate the value of x that minimizes the cost of constructing the bike path.

$C(x) = 400x + 800(x^2-4x+5)^{1/2}$ ⟶ from graph, minimized at $x \approx 1.44$

$C'(x) = 400 + 400(x^2-4x+5)^{-1/2}(2x-4)$

c) Will the value of x that minimizes the cost depend on K? Explain why or why not.

$C'(x) = K + \frac{1}{2}\cdot 2K(x^2-4x+5)^{-1/2}\cdot 2x = 0$ when

$1 + 2x(x^2-4x+5)^{-1/2}$. This does <u>not</u> depend on K.