G.E.R. Lloyd
"Mathematics and Narrative: An Aristotelian Perspective"*

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Overview:

Lloyd takes up the Platonic criticism that the knowledge of mathematical truths is "in direct contradiction with the language" of change and motion used by geometers. For Plato, the objects and truths of mathematics are timeless, pure, and unchanging and as such, the language of activity and motion is not appropriate to them. Lloyd then turns to the Aristotelian perspective on mathematical knowledge. Lloyd proposes that for Aristotle, constructions in geometry amount to the actualization of what is potentially always present in the mathematical objects. As such, no coming into being out of nothing nor passing away is involved in geometry; only the actualization (or in Lloyd's words "revealing") of truths and properties that are always part of the objects *in potentia*.

> "Given that those [geometric] constructions relate to abstract entities, those entities are not altered by the construction being performed: yet what the construction does is to actualize what is there in potentiality, but only in potentiality until the construction is carried out" (Lloyd, 397).

Dates:

Hippocrates	c. 460 - 370 BCE
Plato	427 - 347 BCE
Archytas	428 - 347 BCE
Aristotle	384 - 322 BCE
Plutarch	46 - 120 CE
Proclus	412 - 485 CE

Platonic Position

Famous excerpt from Plato's *Republic* (527ab)

> SOCRATES: This at least will not be disputed by those who have even the slightest acquaintance with geometry, that the branch of knowledge is in direct contradiction with the language used by its adepts.
>
> GLAUCON: How so?
>
> SOCRATES: Their language is most ludicrous, though it cannot help that, for they speak as if they were doing something and as if all their words were directed towards action.

*From Apostolos Doxiadis, Barry Mazur eds. *Circles Disturbed; The Interplay of Mathematics and Narrative.* Princeton: Princeton University Press, 2012: 389 - 40

For all their talk is of "squaring" and "extending" and "adding" and the like, whereas the real object of the entire study is pure knowledge.

GLAUCON: That is absolutely true.

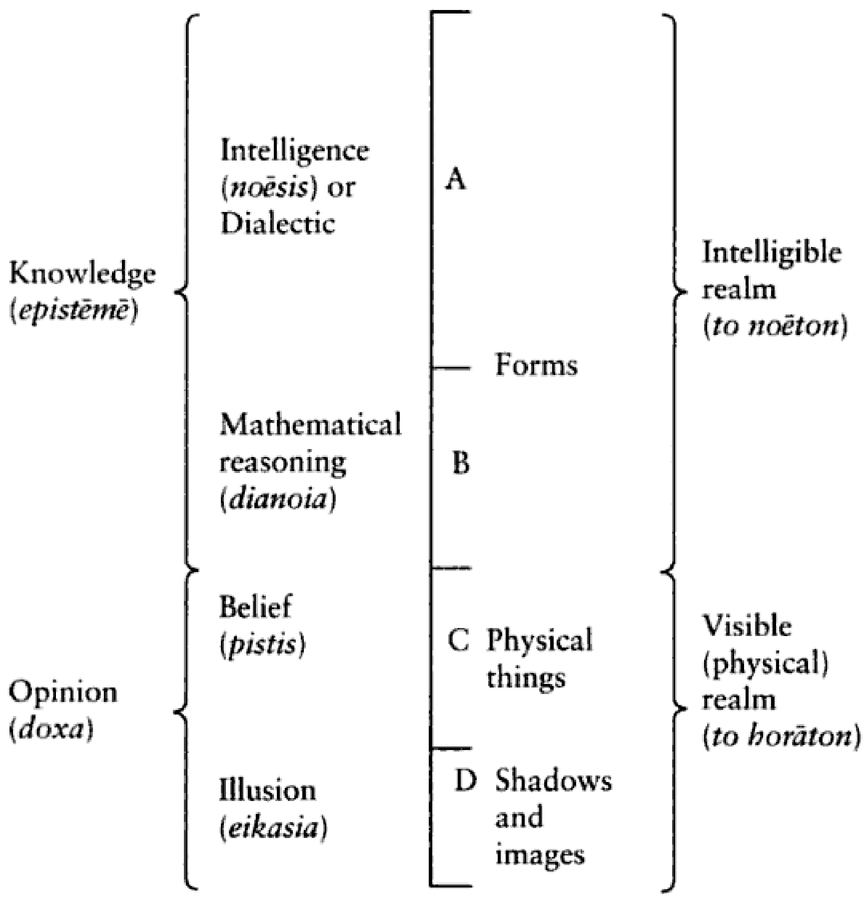
SOCRATES: And must we not agree on a further point?

GLAUCON: What?

SOCRATES: That it is the knowledge of what always is, and not of what at some particular time comes into being and passes away.

GLAUCON: That is readily agreed. For geometry is the knowledge of what always is.[1]

Analogy of the Divided Line: [2]



For Plato there are two orders of reality - one physical and one intelligible. They are accessible to the mind in different ways. The unchanging, immaterial, pure, and timeless "Platonic Forms" occupy the intelligible realm. These Forms can be *known* by the mind which can be trained to dialectically commune with them through thought alone. The Forms are "things in themselves" rather than any particular instance of a thing:

[1] Lloyd adopts his translation of this passage from Plato, *The Republic*, ed. Paul Shorrey. Cambridge, Harvard University Press, 1935: 527ab.

[2] Diagram and interpretation of the "analogy of the Divided Line" taken from Plato, *The Republic*, trans. Desmond Lee. New York: Penguin, 1974: 250.

"And we go on to speak of beauty-in-itself, and goodness-in-itself, and so on for all the sets of particular things which we have regarded as many; and we proceed to posit by contrast a single form, which is unique, in each case, and call it "what really is" each thing".[3]

Mathematical objects are the first of the objects in the intelligible ream "intermediate between intelligible forms and perceptible particulars."[4] "For Plato, mathematical intermediates share **intelligibility** with the forms, but unlike them (but like perceptible particulars) they are **plural**, not singular."[5]

The changing and temporal objects of the material world are like shadows or illusions of those Forms. They are a multiplicity of representations of those singular things-in-themselves. About these shadows, we can only hold beliefs, but not have any true knowledge. This is the realm of motion, change, and activity.

Plato's criticism is of using the language and activity of the material realm to describe mathematical objects which occupy the intelligible realm. That the processes and activities of the material realm should produce *knowledge* of mathematical Forms is a problem for Plato's hierarchy of objects and mental states.

Archytas' Duplication (Doubling) of the Cube:[6]

Lloyd reports that later Platonists would develop and articulate Plato's criticism of motion and activity in mathematics. Plutarch, for example, claims that Plato disapproved of Archytas' method of duplication of the cube - which can't be done by ruler and compass alone.

According to Carl Huffman, Hippocrates showed that the duplication of the cube amounted to the problem of finding two mean proportionals in continued proportion for two given line segments. If we have a segment WX and a segment twice as long YZ then then the cube built with YZ as its side will be 8 times the volume of the cube on side WX, not double. But if two lines can be found, AB and CD such that $WX : AB = AB : CD = CD : YZ$ then the cube built on AB as a side will be double the cube built on WX.

All the terms in the continued proportion are equal to $WX : AB$. If we set them all as such and multiply them together, we get WX^3: AB^3. Multiplying the original terms gives:
$(WX : AB)(AB : CD) = WX : CD$, $(WX : CD)(CD : YZ) = WX : YZ$.
Therefore WX^3: $AB^3 = WX : YZ$
But we know that YZ is double WX.
Therefore we know that the cube AB^3 is double the cube WX^3 as desired.

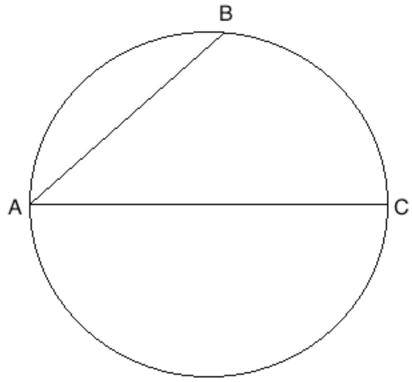
Archytas does not cite Hippocrates and doesn't even formulate his solution in terms of duplicating the cube. Instead, he sets out just to find two mean proportionals in continued proportion for two given line segments. Huffman thinks Archytas is picking up where Hippocrates left off.[7]

Heath: "The solution of Archytas [to the duplication of the cube] is the most remarkable of all, especially when his date is considered (first half of fourth century B.C.), because it is not a construction in a plane but a bold construction in three dimensions, **determining a certain point as the intersection of three surfaces of revolution.**"[8]

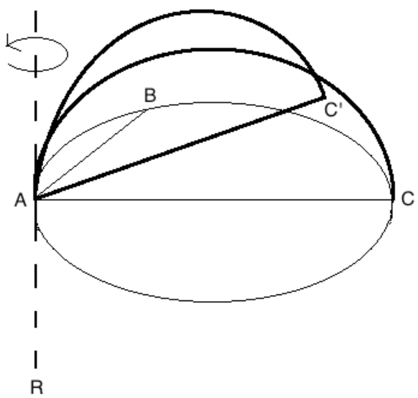
[3]Plato, *The Republic*, trans. Desmond Lee: 507b. There are many different translations of this passage.
[4]Lloyd, 392.
[5]Lloyd, n.7
[6]From Thomas Heath, *A History of Greek Mathematics* Vol. 1. Oxford: Oxford University Press, 1921: 246 - 249
[7]Demonstration is from from Carl Huffman, *Archytas of Tarentum; Pythagorean, Philosopher, and Mathematician King*. Cambridge: Cambridge University Press, 2005: 350.
[8]Heath, *A History of Greek Mathematics* Vol. 1: 247, my emphasis.

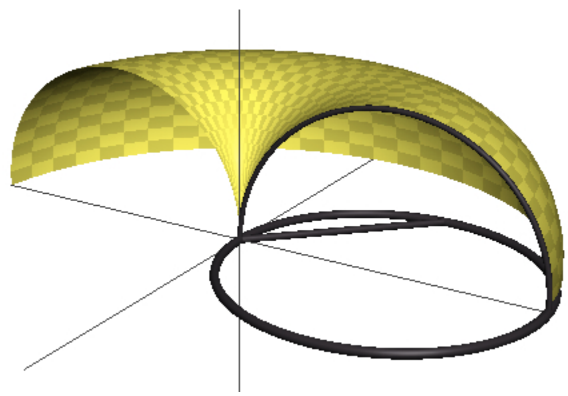
Suppose AC and AB are two straight lines between which two mean proportionals are to be found.

Let AC be the diameter of a circle and AB be a chord in that circle.

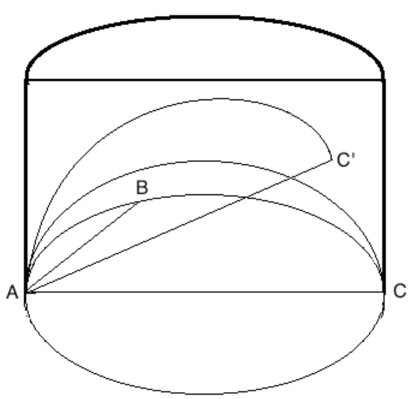


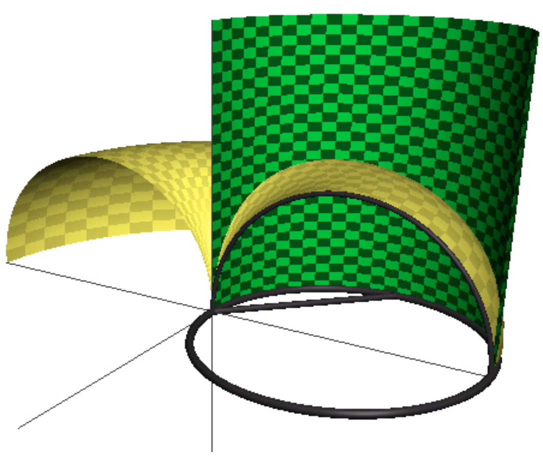
Draw a semicircle with diameter AC but in a plane perpendicular to the the plane of circle ABC. **Imagine this semicircle to revolve about a straight line R that runs through point A and is perpendicular to the plane of circle ABC.** This rotation describes a half-torus whose inner diameter is *nil*.



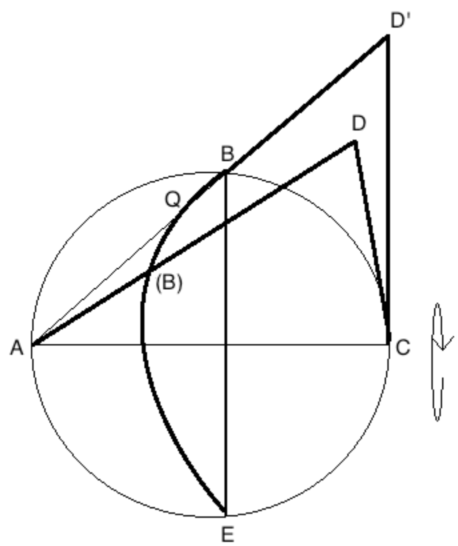


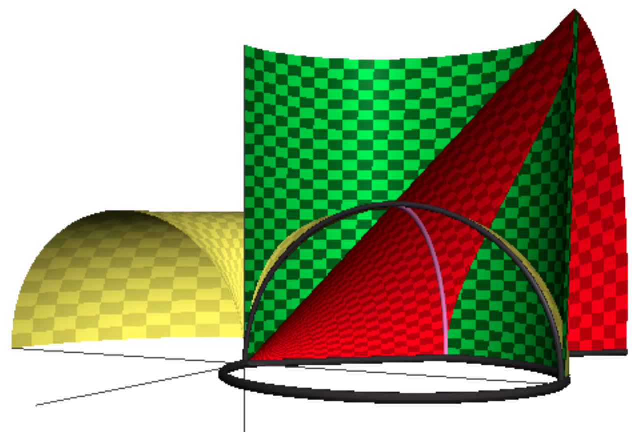
Next, draw a half-cylinder on the semicircle ABC as base. This half-cylinder will cut the half-torus in a certain curve. [All 3D images courtesy of Travis Dick].



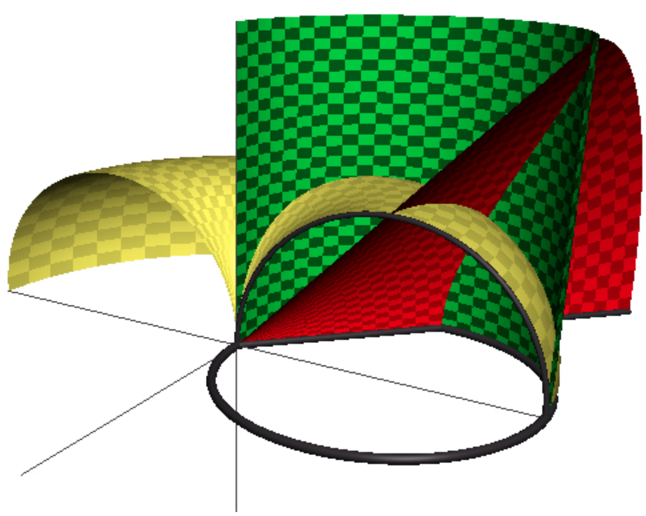


Finally, let CD, the tangent to the circle ABC at point C meet AB extended by D to form a right triangle ACD. **Suppose that triangle ACD rotates around axis AC.** This rotation describes the surface of a right circular cone (D, D'. As the triangle rotates, the point B will describe a semicircle BQE at right angles to the plane ABC and having its diameter BE at right angles to AC. (cylinder and semicircle on AC are removed below for clarity)





The surface of the cone described by the rotation of ACD will meet the curve created by the intersection of the half-cylinder and the half-torus (described by the rotating half-circle on AC) at a point P. Put another way, P is the point at which the surface of the cone intersects the curve created by the intersection of the half-cylinder and the half-torus.



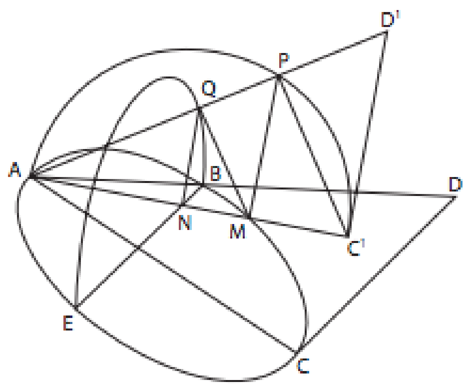
Because P is a point on the half-torus, it can be understood as on the circumference of the revolving semicircle on AC'.

Let APC' be the position of the revolving semicircle (diameter AC' revolving around R) when it is in position to intersect the cylinder and the surface of the cone, and let AC' meet the circumference of circle ABC at point M. [Freeze the revolving semicircle that describes the torus at the point P where all intersect. Once we have P we don't need the torus anymore, we need the semicircle with point P on the circumference which gives the rest of the points].

Not only is point P on the circumference of the revolving semicircle, it is also on the surface of the cylinder that stands on half-circle ABC. Draw a line PM perpendicular to the plane of ABC. We know that point M will be on the circumference of circle ABC because P is on the cylinder.

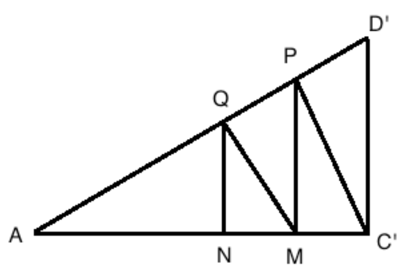
But point P *also* lies on the surface of the cone created by the rotating triangle ACD. Recall that point B imagined to rotate with that triangle described a semicircle with diameter BE. Let Q be the point on the circumference of that semicircle that is in line with A and P. Let line segment AP meet the circumference of the semicircle BQE at point Q.

Recall that AC' is the diameter of the revolving semicircle that we have now frozen at point P. That line is in the same plane as the original circle ABC and is also in the same plane as the diameter of the semicircle BQE that was created by point B imagined in rotation with triangle ACD. Let that line segment AC' meet the diameter of that semicircle (BQE) at point N.



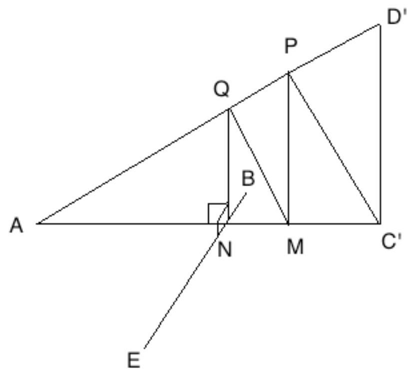
(Figure from Heath, *A History of Greek Mathematics* Vol. 1: 247)

Join PC', QM, QN.

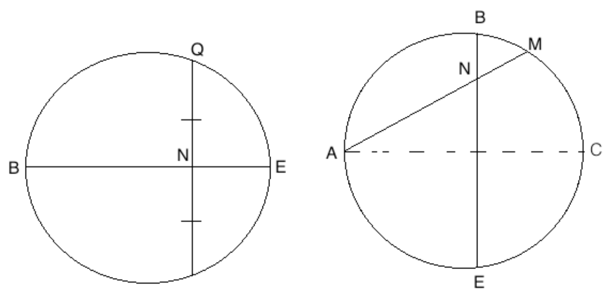


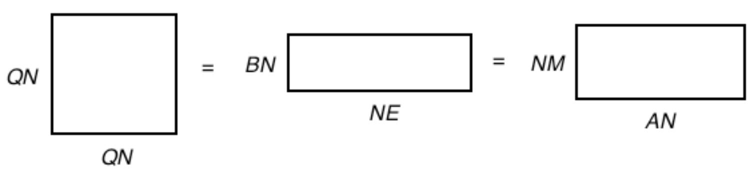
QN is perpendicular to the plane ABC because it is the line of intersection of half-circles APC' and BQE which are both perpendicular to that plane [by Euclid XI. 19: "If two planes which cut one another are at right angles to any plane, then their intersection is also at right angles to the same plane".]

Therefore QN is perpendicular to BE which is on the plane ABC.



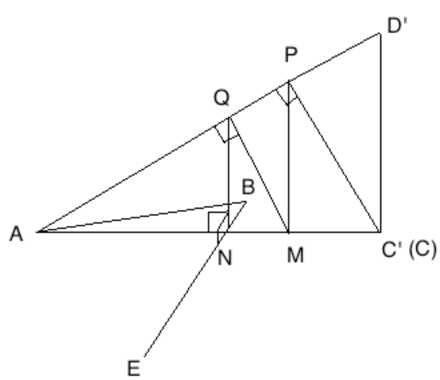
Therefore $QN^2 = BN\ .\ NE = AN\ .\ NM$ [By Euclid III. 35: "If in a circle two straight lines cut one another, then the rectangle contained by the segments of the one equals the rectangle contained by the segments of the other". Here, there are two sets. *2QN* would cross *BNE* in the circle *BQE*. *ANM* crosses the *BNE* in the perpendicular circle *AEC*. The rectangles contained by the segments of those two sets of intersecting lines are therefore all equal.]





so that the angle AQM is a right angle.

But the angle *APC'* is also a right angle; therefore, MQ is parallel to *C'P*.



Triangles AQM and *APC'* are similar. It follows, by similar triangles, that

$AC' : AP = AP : AM = AM : AQ;$

that is, $AC : AP = AP : AM = AM : AB$, [we can replace Q with B here because Q is just an instance of B imagined to revolve with the triangle ACD and we can replace C' with C because C' is just C imagined to rotate with the diameter of the revolving semicircle]

or $AB : AM = AM : AP = AP : AC$ [reverse order to resemble Hippocrates construction]

and AB, AM, AP, AC are in continued proportion **so AM and AP are the two mean proportionals required.**
For example, where $AC = 2AB$, the demonstration from Hippocrates given above yields that the cube with side AM will be double the cube with side AB.

In response to this demonstration, Lloyd reports: **"Plutarch has it that Plato "was angry" with Archytas and Eudoxus and "inveighed against them as the destroyers and corrupters of what is good in geometry, which thus ran away from incorporeal and intelligible things and made use of bodies that required much laborious work."**[9]

Aristotelian Perspective:

1. *The ontology of mathematical objects*: According to Lloyd, Aristotle "did not postulate separate intelligible mathematical objects" [or of any Forms] that occupied some intelligible realm independent of the material world.[10] Instead, for Aristotle, mathematics is the study of mathematical properties - like "circularity" - of physical objects - like rings and hoops - in the abstract. "Circularity" is not an entity in itself - a Form of the thing-in-itself - as Plato would have it. It is a property of physical objects that can be studied in the abstract.

2. *The status of activity and motion in mathematics*: Aristotle does not consider the activity of mathematicians and their use of construction and motion to be in tension with the nature of mathematical knowledge. They don't, in Aristotle's view, *change* anything about the mathematical objects by those activities. Instead, they *actualize* [*energeia*] (in Lloyd's words "reveal", "make manifest") what is always already present *potentially* in those objects.

   "*diagrammata* too in mathematics are discovered by an actualization [*energeia*], for it is by a process of dividing them up that they [the mathematicians] discover them. If the division had already been performed, they [the *diagrammata*] would have been manifest: as it is, they are present only potentially... Hence it is manifest that relations subsisting potentially are discovered by being brought into actuality [*energeia*]: the reason is that the exercise of thought is an actuality [*energeia*]."[11]

Translational Issues:

- *energeia*: Can mean both the "activity" of the mathematicians and "actualization" or "bringing into actuality" - especially of mathematical properties that "come to light" through the divisions and constructions that the mathematicians do. Lloyd thinks there is some slippage in Aristotle, and not all translators agree, but he prefers the meaning of "actuality" or "bringing into actualization". The word *energeia* is contrasted with the word *kinesis* for "movement" or "change". This is important as Aristotle, like Plato, does not think that mathematical proofs move or chance anything. Rather, proofs are discovered through actualization - and getting at what this means is the heart of Lloyd's project.

- *diagrammata*: Lloyd leaves this untranslated in his account. Can mean "diagram". Can also mean "geometric propositions" including their "proofs". ["If the division had already been performed,

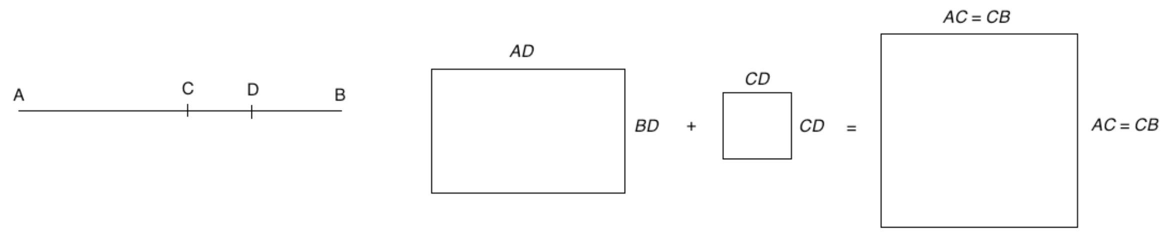
[9] Quoted in Lloyd, 391.
[10] Lloyd, 392.
[11] Quoted in Lloyd, 392.

they [the *diagrammata*] would have been manifest". Lloyd thinks that surely he means here that the *proof* would have been manifest, not the construction itself.]

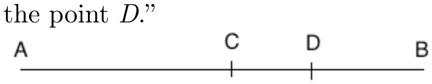
energeia in Action: Proclus on Geometric Reasoning (*Elements* II.5)

In the remainder of the paper, Lloyd looks for evidence of Aristotle's position on mathematical proof as actualization in the *practices* of Greek geometers.

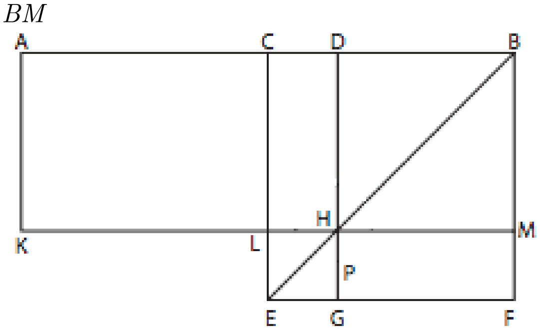
In 5th century CE Proclus outlined six steps of geometric reasoning in which Lloyd thinks the idea of actualization through construction can be identified. Reviel Netz gives an example of these six steps from Euclid, II.5: "If a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole, with the square on the line between the cuts, is equal to the square on the half"



1. *protasis*: enunciation of the proposition to be proved
   "If a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole, with the square on the line between the cuts, is equal to the square of the half".

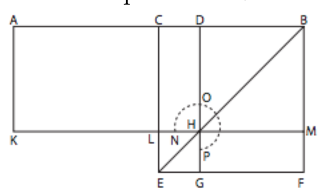
2. *ekthesis*: setting out
   "For let some line, AB, be cut into equal segments at the point C and at unequal segments at the point D."
   

3. *diorismos*: the definition of the goal, specification of the conditions for reaching the goal
   "I say that the rectangle contained by AD and DB together with the square on CD, is equal to the square on CB."

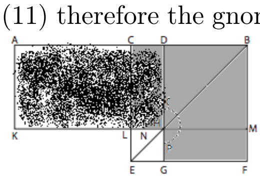
4. *kataskeuē*: the construction
   (a) For on CB, let a square be set up, namely $CEFB$
   (b) and let the line BE be joined
   (c) and, through the point D let the line DG be drawn parallel to either of the lines CE, BF
   (d) and, through the point H, again let the line KM be drawn parallel to either of the lines AB, EF
   (e) and again, through the point A, the the line AK be drawn parallel to either of the lines CL, BM
   

Lloyd proposes that it is *this diagram* that will "enable the proof to proceed. Before that construction is effected, the reader will be at a loss as to how the proposition can be shown. Merely inspecting the diagram will not be enough to give the proof, since certain equalities must be established, directly or by appealing to earlier results. But the proof depends on reasoning carried out on the figures as constructed, not on them as originally given in the enunciation".[12]

5. *apodeixis*: the proof proper (based on the construction), taken here mainly from Netz
(1) The complement *CH* is equal to the complement *HF* [by *Elements* I.43: "In any parallelogram [here *CEFB*], the complements of the parallelograms about the diameter equal one another". So in this case, *CH* and *HF* are the complements of the parallelogram *LHGE* about the diameter *BE*]
(2) Let the square *DHMB* be added to both complements, which will remain equal
(3) therefore, the whole area of *CLMB* is equal to *DGFB*
(4) but we know that the area of *CLMB* is equal to the area of *AKLC*.
(5) since the line *AC* is equal to the line *CB*
(6) therefore, the area of *AKLC* is equal to the area of *DGFB*.
(7) Let the area of *CLHD* be added as common
(8) therefore the whole *AKHD* is eqal to the gnomon *NOP*. [*Elements*, definition II.2: "And in any parallelogram area let any one whatever of the parallelograms about the diameter with the two complements be called a *gnomon*." Here the gnomon consists of parallelogram *DHMB* and its two complements *GHMF* and *HLCD*].



(9) But the area *AKHD* is the rectangle contained by the lines *AD* and *DB*
(10) for the line *DH* is equal to the line *DB*
(11) therefore the gnomon *NOP* too is equal to the rectangle contained by the lines *AD* and *DB*



(12) Let the are of *LEGH* be added as common
(13) which is equal to the square on *CD* [which is the "line between cuts" from the original enunciation]
(14) therefore the gnomon *NOP* and the area *LEGH* are equal to the rectangle contained by *AD* and *DB* plus the square on the line between the cuts, built on *CD*
(15) but the gnomon *NOP* and the area *LEGH* taken together is the square on *CB*
(16) therefore, the rectangle contained by the lines *AD* and *DB* plus the square on *CD* is equal to the square on the line *CB*.

6. *superasma*: the conclusion
Therefore if a straight line is cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole, with the square on the line between the cuts, is equal to the square on the half; which it was required to prove.

"Two points stand out as fundamental. First, the steps are sequential and the theorem as a whole involves what Aristotle would call an *energeia*, the actualization of a certain potentiality, and indeed the activity of the mathematical reasoner to bring that about. Second,

[12]Lloyd, 399.

> the proof depends crucially on the construction... [T]he proof depends on reasoning carried out on the figures as constructed, not on them as originally given in the enunciation."[13]

Other examples of proofs that depend upon a *sequence* of steps, an *energeia*

- Proof of the Pythagorean Theorem (*Elements* I.47)
- Archytas' Duplication of the Cube (from above)
  "Yet there could be no more striking example mathematical reasoning depending on constructions involving surfaces and solids in revolution, the generation of a curve, and the identification of a point by considering the intersection of that curve and the cone".[14]
  So where, for Plato, mathematics was an important part of mental training for how to commune with the Forms independent of the material world, the *reasoning* involved in these demonstrates is utterly dependent on the constructions. According to Lloyd, it is the diagrams that give access to the cognitive paths that lead to the truths that are, for both Plato and Aristotle, timeless and unchanging.
- Greek arithmetic, including the proof of the infinity of primes (*Elements* IX.20)
- Greek mathematical iterative processes, including those that proceed indefinitely
- Chinese mathematics (procedures carried out on counting boards)

Temporality and Narrative:

Lloyd is also interested (as are all the articles in the volume) with the relationship between narrative and mathematics. Narrative, says Lloyd, deals with the chronological recounting of sequences of events. Are geometric proofs narrative in nature? Mathematical truths and objects are timeless. But mathematical proofs "take time to set out, to be actualized".[15] However, unlike stories or walking trips to Athens, "any *energeia* is complete at any moment of time."[16] Proofs are not complete in the sense that they are not understood by the mathematician until the construction is done and the necessary reasoning carried out upon it, but the truth in question is complete the whole way along.

> "In mathematical reasoning, time in the sense of chronology is not relevant, since the truths revealed are indeed timeless. On the other hand, the reasoning does involve a sequence of steps that are essential to reveal, or as Aristotle would say to actualize, the truths that are there in potentiality in the geometric figures or the quantities discussed."[17]

[13] Lloyd, 399.
[14] Lloyd, 400 - 401.
[15] Lloyd, 397.
[16] Lloyd, 397.
[17] Lloyd, 403.

Concluding Questions:

- Although Lloyd begins with the Platonic criticism (through Plutarch) of Archytas' duplication of the sphere by *rotating* surfaces, he then turns to focus on the idea of *sequence* in mathematical constructions and proofs. Do constructions involving motion present a unique case of *energeia* or are they the same in kind as the latter construction from Euclid that is a traditional two-dimensional ruler and compass construction?

- Lloyd seems to, at least implicitly, focus a lot of attention on *the mathematician.* Mathematicians are the *actualizers.* They, by their reasoning power, enact the *energeia* in these proofs. He also seems to focus somewhat on the *understanding* of the mathematician. Even though the truths of mathematics are timeless for Aristotle, the mathematicians must follow a sequence of steps to come to understand them. Does thinking about the role of the mathematician in proof and construction add anything to our question about the role that motion and mechanics play?

- In the passage from *The Republic* that Lloyd quotes at the beginning of the article, Socrates says the following: "Their language is most ludicrous, *though it cannot help that,* for they speak as if they were doing something and as if all their words were directed towards action. For all their talk is of "squaring" and "extending" and "adding" and the like, whereas the real object of the entire study is pure knowledge."[18] In this statement, Plato (through Socrates) seems to suggest that in spite of the disjoint between the language geometers use and the objects of their knowledge, there is something *necessary* about that language - "it cannot help that". Might this suggest that the Platonic and Aristotelian perspectives are not completely opposed?

- Lloyd makes a kind of implicit distinction between using motion in geometric construction and using actual imagined physical devices like Archimedes does. Would Aristotle admit physical mechanical devices into the sequences through which an *energeia* happens?

[18]Lloyd, 389, my emphasis.