Math 25a Homework 3 Solutions part 2

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1 Examples of Metric Spaces

2 Metric spaces

(1) (a) Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^k$. Find the value of t that minimizes $|\vec{x} + t\vec{y}|$. (So, just to be very clear, I'm using some notation here to distinguish between $x \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^k$. That is $\vec{x} = (x_1, x_2, \ldots, x_k)$, where each $x_i \in \mathbb{R}$.)

(b) Deduce that $|\vec{x} \cdot \vec{y}| \le |\vec{x}||\vec{y}|$ for all $\vec{x}$, $\vec{y} \in \mathbb{R}^k$.

(c) Prove the *triangle inequality:* $|\vec{x} + \vec{y}| \le |\vec{x}| + |\vec{y}|$ for all $\vec{x}, \vec{y} \in \mathbb{R}^k$.

(d) Hence or otherwise, show that if we define $d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|$, then $(\mathbb{R}^k, d)$ is a metric space.

Hint: The definitions of $\vec{x} \cdot \vec{y}$ and $|\vec{x}|$ are on page 16 of Rudin.

Solution.

(a) Let $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. Then, noting that $|x + ty|$ is non-negative, minimizing it is equivalent to minimizing the square of it:

$$0 \le |x + ty|^2 = \sum_{i=1}^{k} (x_i + ty_i)^2 = t^2 \left[\sum_{i=1}^{k} y_i^2 \right] + 2t \left[\sum_{i=1}^{k} x_i y_i \right] + \sum_{i=1}^{k} x_i^2$$
$$= A \left(t + \frac{B}{2A} \right)^2 + C - \frac{B^2}{4A}$$

Where A, B, C are the coefficients of X above.

So, $|x + ty|$ is minimized for $t = -\frac{B}{2A} = -\frac{\sum_{i=1}^{k} x_i y_i}{\sum_{i=1}^{k} y_i^2}$.

(b) We note that the LHS in our expression in (a) was always positive, and so the RHS must be as well. This yields:

$$\frac{4AC - B^2}{4A} \ge 0 \Rightarrow \left[\sum_{i=1}^{k} y_i^2 \right] \left[\sum_{i=1}^{k} x_i^2 \right] \ge \left[\sum_{i=1}^{k} x_i y_i \right]^2 \Rightarrow |x|^2 |y|^2 \ge (x \cdot y)^2 \Rightarrow |x||y| \ge |x \cdot y|$$

(c) Let x, y be as in (a):

$$|x + y|^2 = |x|^2 + |y|^2 + 2(x \cdot y) \le |x|^2 + |y|^2 + 2|x \cdot y| \le |x|^2 + |y|^2 + 2|x||y| = (|x| + |y|)^2$$

So, as $x \mapsto x^2$ is strictly increasingon the non-negative reals, we have $|x+y|^2 \le (|x|+|y|)^2$. But, as both $|x+y|$ and $|x|+y|$ are non-negative quantities, this shows $|x+y| \ge |x|+|y|$, as desired. □

Now, $d(x,y) = |x-y|$ is a distance, making $(\mathbf{R}^k, d)$ a metric space as d has:

(a) *Positivity*: For $x \neq y$ (so some coordinate differs), $d(x,y) = |x-y| = \sqrt{\sum_{i=1}^k (x_i - y_i)^2} > 0$, and $d(x,x) = |x-x| = \sqrt{\sum_{i=1}^k 0} = 0$.

(b) *Symmetry*: We have that $d(x,y) = |x-y| = \sqrt{\sum_{i=1}^k (x_i - y_i)^2} = \sqrt{\sum_{i=1}^k (y_i - x_i)^2} = |y-x| = d(y,x)$.

(c) *Triangle Inequality*: By the result above, $d(p,q) = |p-q| = |(p-r)+(r-q)| \le |p-r|+|r-q| = d(p,r)+d(r,q)$.

(2) Problem 10 on page 44 of Rudin.

Solution. We verify that it satisfies the properties of a metric:

(a) *Positivity*: For $p \neq q$ we have $d(p,q) = 1 > 0$, and we have $d(p,p) = 0$.

(b) *Symmetry*: For $p \neq q$ we have $d(p,q) = 1 = d(q,p)$, and symmetry is trivial when $p = q$.

(c) *Triangle Inequality*: For $p, q, s \in X$, either they are all the same, or some one differs. If they are all the same, then we get $d(p,s) = 0 \le 0+0 = d(p,q)+d(q,s)$, and the triangle inequality holds. If one differs, then the LHS will be at most 1, while the RHS will be at least 1, and the triange inequality again holds.

In the topology given by this metric, any set of points is open (as any neighborhood of radius less than 1 will be contained in the set). So, it follows that any set of points is closed (just taking the complement in X). Any finite subset is cleary compact (from any open cover, take one set containing each point, thus clearly we have a finite subcover). Infinite subsets are not compact, for we could have an open cover consisting of just a set containing each point, and then there is no proper subcover, and hence no finite subcover. □

(3) Problem 11 on page 44 of Rudin.

Solution. We note that d_1 *is not* a metric, for it doesn't satisfy the triangle inequality: $d_1(-1,1) = 4 \not\le 2 = 1+1 = d_1(-1,0)+d_1(0,1)$.

We note that d_2 *is* a metric, for it clearly satisfies non-negativity and symmetry. For the triangle inequality, we note that $x \mapsto \sqrt{x}$ is strictly increasing on the non-negative reals, and check:

$$\begin{aligned} d_2(p,s) &= \sqrt{|p-s|} \le \sqrt{|p-q|+|q-s|} \le \sqrt{|p-q|+|q-s|+2\sqrt{|p-q||q-s|}} \\ &= \sqrt{|p-q|} + \sqrt{|q-s|} = d_2(p,q)+d_2(q,s) \end{aligned}$$

We note that d_3 *is not* a metric, for it doesn't satisfy non-negativity: $d_3(1,-1) = 0$, and $1 \neq -1$.

We note that d_4 *is not* a metric, for it doesn't satisfy non-negativity: $d_4(1,1) = |1-2| = 1 \neq 0$.

We note that d_5 *is* a metric. We use the fact that d from (10) was a metric. Now, d_5 clearly satisfies $d_5(p,q) > 0$, and $d_5(p,p) = 0$. Also, as d is symmetric, d_5 must be as well. Finally, we check the triangle inequality:

$$\begin{aligned} d_5(p,s) &= \frac{|p-s|}{1+|p-s|} = 1 - \frac{1}{1+|p-s|} \leq 1 - \frac{1}{1+|p-q|+|q-s|} \\ &= \frac{|p-q|+|q-s|}{1+|p-q|+|q-s|} \leq \frac{|p-q|}{1+|p-q|} + \frac{|q-s|}{1+|q-s|} = d_5(p,q) + d_5(q,s) \end{aligned}$$

□

(4) *This problem introduces the p-adic topology on* $\mathbb{Q}$. *In thinking about this quesiton, it might be helpful to think of a particular p, the 3-adic topology for example.*
Given a non-zero rational number r, we can write it uniquely in the form

$$r = \frac{p^\nu n}{d}$$

where n and ν are integers, d is a positive integer, and neither n nor d is divisible by p. Define $\nu(r)$ to be the integer ν occurring in this expression. For $x, y \in \mathbb{Q}$, define

$$d_p(x,y) = \begin{cases} p^{-\nu(x-y)} & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

(a) Show that $(\mathbb{Q}, d_p)$ is a metric space, and that in fact $d(x,z) \leq \max(d(x,y), d(y,z))$.

(b) Show that if $x \in N_r(a)$, then $N_r(x) = N_r(a)$, so that any point of the neighborhood $N_r(a)$ is a "center" of that neighborhood.

(c) Show that given two neighborhoods $N_{r_1}(a_1)$ and $N_{r_2}(a_2)$, either they are disjoint or one is contained in the other.

This metric space is weird huh?

Solution. (a) As usual, we have three properties to check:

(a) *Positivity*: For $p \neq q$ we have $d_p(p,q) = p^{-\nu(x,y)} > 0$, and we have $d(p,p) = 0$.

(b) *Symmetry*: For $p \neq q$, we can write $x - y = \frac{p^{\nu(x,y)}n}{d}$, so $y - x = \frac{p^{\nu(x,y)}(-n)}{d}$: because representations of this form are unique, $\nu(y-x) = \nu(x-y)$. Hence $d_p(x,y) = d_p(y,x)$.

(c) *(Stronger) Triangle Inequality*: If any two of the three points x, y, z are equal, the triangle inequality becomes trivial (even in its stronger form). Otherwise, for $x, y, z \in \mathbb{Q}$, write

$$x - y = \frac{p^{\nu(x-y)}n_1}{d_1}, y - z = \frac{p^{\nu(y-z)}n_2}{d_2}$$

Then

$$x - z = (x-y) + (y-z) = \frac{p^{\nu(x-y)}n_1}{d_1} + \frac{p^{\nu(y-z)}n_2}{d_3} = \frac{p^{\nu(x-y)}n_1 d_2 + p^{\nu(y-z)}n_2 d_1}{d_1 d_2}$$

The numerator of this fraction is divisible by $p^{\min(\nu(x-y),\nu(y-z)}$, so can be written in the form $p^{\nu^*}n^*$, where $\nu^* \geq \min(\nu(x-y), \nu(y-z)$ and p does not divide n^*. That is,

$$x - z = \frac{p^{\nu^*}n^*}{d_1 d_2},$$

and p divides neither n^* or d_1d_2, so we can apply the definition of ν to get $\nu(x-z) = \nu^* \geq \min(\nu(x-y), \nu(y-z)$. Applying the decreasing function $m \mapsto p^{-m}$,

$$d(x,z) \leq p^{-\nu(x-y)} \leq p^{-\min(\nu(x-y),\nu(y-z)} = \max(d(x,y), d(y,z)).$$

This is the stronger form of the triangle inequality that we were asked to prove. It implies the triangle inequality because $d(x,y) + d(y,z) \geq \max(d(x,y), d(y,z))$.

Note: Many of you tried to prove that $\nu(x-y) = \min \nu(x-y), \nu(y-z)$. This is true when $\nu(x-y) \neq \nu(y-z)$ – however, if $\nu(x-y) = \nu(y-z)$, one can pull extra factors of p out of the numerator of $x-z$ when p divides $n_1d_2 + n_2d_1$. The fact that $\nu(x-y) = \min \nu(x-y), \nu(y-z)$ in many cases is still an interesting result: for one thing, it means that all triangles are isosceles in the p-adic metric on $\mathbb{Q}$ (contrast with the normal metric on $\mathbb{Q}$, where all triangles are degenerate).

b) We first show that $N_r(x) \subset N_r(a)$. Let y be any element of $N_r(x)$. Then using our stronger version of the triangle inequality from a), $d_p(y,a) \leq \max(d_p(y,x) + d_p(x,a)) < r$, because a, y both lie in $N_r(x)$. So by definition of $N_r(a)$, $y \in N_r(a)$ as well. Because y could be any element of $N_r(x)$, $N_r(x) \subset N_r(a)$. The exact same argument with x, a switched shows $N_r(a) \subset N_r(x)$. So the two sets are equal.

c) Suppose that $N_{r_1}(a_1)$ and $N_{r_2}(a_2)$ are not disjoint. Then let x be a point in their intersection. By part b), $N_{r_1}(a_1) = N_{r_1}(x)$ and $N_{r_2}(a_2) = N_{r_2}(x)$. Assume without loss of generality that $r_1 \leq r_2$. Then for any $y \in N_{r_1}(x)$, $d(y,x) < r_1 \leq r_2$, so $y \in N_{r_2}(x)$ as well. That means that $N_{r_1}(a_1) = N_{r_1}(x) \subset N_{r_2}(x) = N_{r_2}(a_2)$.

□

3 More open sets, closed sets

(3) Problem 9 parts (a) through (d) on page 43 of Rudin.

Solution.

(a) $p \in E^\circ \Rightarrow \exists N_r(p) \subset E \Rightarrow N_r(p) \subset E^\circ \Rightarrow E^\circ$ is open.

(b) If $E^\circ = E$, since E° is open, E is also open. If E is open, then $E \subset E^\circ$. But since $E^\circ \subset E$, $E^\circ = E$.

(c) $p \in G \Rightarrow p \in N_r(p) \subset G$ for some $r \Rightarrow p \in E \Rightarrow p$ is an interior point $\Rightarrow p \in E^\circ$.

(d) Suppose p is in the complement of E°. Then either $p \in E^c$ or $p \in E$ and $N_r(p)$ is not in E. In the first case, $p \in E^c \subset \overline{E}^c$. In the second, for each N_rp, there must be a point in it that is not in E, and thus in E^c. So p is a limit point of E^c, and is thus in its closure.

For the other direction, suppose $p \in \overline{E^c}$. Either $p \in E^c$ or p is a limit point of E^c and in E. In the first case $p \in E^c \subset (E^o)^c$. In the second place p cannot be an interior point of E since this means for some r, $N_r(p)$ is completely in E and thus p cannot be a limit point of E^c, so $p \in (E^o)^c$.

□

(4) Problem 29 on page 45 of Rudin.

Solution. (a) Consider $0 \subset \mathbf{R}$, open. From the separability for $\mathbf{R}$ (such as by $\mathbf{Q}$) there is a countable dense subset. Consider a surjection $\mathbf{N} \to O \cap \mathbf{Q}$ and let p_i be the image of i under the surjection. Then since O is open, if $x \in O$ then $(x-\delta, x+\delta) \subset O$ for some $\delta > 0$ must contain at least one of the p_i. Consider successively each of the p_i. For each one there is an interval $(p_i - \epsilon, p_i + \epsilon) \subset \mathbf{Q}$. Now if $[p_i, \infty) \nsubseteq$ consider

$$A_i = \sup\{t \in \mathbf{R} | [p_i, t) \subset O\}.$$

In the same way if $(-\infty, p_i] \nsubseteq O$ then consider

$$B_i = \sup\{t \in \mathbf{R} | [p_i, p_i - t) \subset O\}.$$

In all cases we obtain an open interval $(p_i - B_i, p_i + A_i) \subset O$, possibly infinite in one direction or the other. By definition this interval is maximal in O and contains p_i. In other words this interval is the union of all intervals in O containing p_i. Now dropping the p_i which are contained in the same intervals (formally we can go to equivalence relations) we find that we have at most a countable number of p_i remaining and hence a countable number of disjoint intervals with a union equal to O.

□

4 Open Sets, Closed Sets

(1) Problem 6 on page 43 of Rudin.

Solution.

(a) Let $x_0 \in E''$. Fix $r > 0$. As x_0 is a limit point of E', we have that $(N_{\frac{r}{2}}(x_0) \setminus \{x_0\}) \cap E' \neq \emptyset$. Let y be an element of the intersection. Let $r' = d(x_0, y) > 0$. Then, as $y \in E'$ we must have $(N_{r'}(y) \setminus \{y\}) \cap E \neq \emptyset$; let t be an element of the intersection. So, $d(x_0, t) \leq d(x_0, y) + d(y, t) < r$, and $x_0 \neq t$ (for $d(t, y) < r' = d(x_0, y)$). So, $(N_r(x_0) \setminus x_0) \cap E \neq \emptyset$. As this holds for each $r > 0$, we have that x_0 is a limit point of E.

(b) As $E \subset \bar{E}$, any limit point of E must be a limit point of $\bar{E}$, so $E' \subset \bar{E}'$. We claim that for any $x_0 \in \bar{E}'$, $x_0 \in E'$ (so the inclusion holds in the other direction as well): Fix $r > 0$. Now, let $y \in (N_{\frac{r}{2}}(x_0) \setminus \{x_0\}) \cap \bar{E}$ (which exists as x_0 is a limit point of $\bar{E}$). If $y \in E$, then $y \in (N_r(x_0) \setminus \{x_0\}) \cap E$. If $y \in E' \setminus E$, any punctured neighborhood of y has non-empty intersection with E, so in particular there is a $t \in N_{\frac{r}{2}}(y) \cap E$ (note that we have $t \neq x$). Then, $d(x_0, t) \leq d(x_0, y) + d(y, t) < \frac{r}{2} + \frac{r}{2} = r$, so $t \in (N_r(x_0) \setminus \{x_0\}) \cap E$. In either case, $(N_r(x_0) \setminus \{x_0\}) \cap E \neq \emptyset$. As this holds for each $r > 0$, we have that x_0 is a limit point of E.

(c) Let $E = \{1 - \frac{1}{n} | n \in \mathbf{N}_{n>0}\}$. Then, $E' = \{1\}$, so then $E'' = \emptyset$. □

(2) Problem 10 on page 44 of Rudin.

Solution.

(a) Let X, Y be sets, and $S = X \cup Y$. Then, we claim $\bar{S} = \bar{X} \cup \bar{Y}$. We note immediately that any limit point of X or Y is also a limit point of $X \cup Y$ (non-empty intersections will remain so!), so $\bar{S} \supset \bar{X} \cup \bar{Y}$. Now, let $x_0 \in \bar{S}$. For each $n \in \mathbf{N}_{>0}$, let $x_n \in N_{\frac{1}{n}}(x_0) \setminus \{x_0\}$. Then, take the subsequence $\{x_{a_1}, x_{a_2}, x_{a_3}, \ldots\}$ $(a_1 < a_2 < \cdots)$ of elements of $\{x_1, x_2, \ldots\}$ that lie in X, and $\{x_{b_1}, x_{b_2}, \ldots\}$ $(b_1 < b_2 < \cdots)$ that lie in Y. These two sequences must contain all elements of $\{x_i\}$ (for $S = X \cup Y$), and so can not both be finite. Without loss of generality, say the sequence of elements in X is infinite. Then, for any $r > 0$, take $n > \frac{1}{r}$, and note that $a_n \geq n$, so $x_{a_n} \in N_{\frac{1}{n}}(x_0) \setminus \{x_0\} \subset N_r(x_0) \setminus \{x_0\}$. So, $x_0 \in \bar{X}$ (the apparent lack of symmetry here is from our "without loss of generality," and is thus only apparent). This shows the inclusion the other way, proving our claim.

(b) We elaborate on the claim we dismissed as trivial in (a): Let $x \in \bar{A}_i$. Then, for each $r > 0$, there exists a $x' \in A_i$, $x' \neq x$ with $d(x, x') < r$. Now, $A_i \subset B$, and thus $x' \in B$. So, $x \in \bar{B}$, proving the desired inclusion. For the example, let $A_i = \{1 - \frac{1}{i}\}$. Then, $\bar{A}_i = A_i$, so $\bigcup_{i=1}^{\infty} \bar{A}_i = \bigcup_{i=1}^{\infty} A_i = \{1 - \frac{1}{i} | i \in \mathbf{N}_{>0}\}$. But, $\bar{B} = B \cap \{1\}$.

□

5 Compactness

(1) Problem 12 on page 44 of Rudin.

Solution. Suppose we have an open cover. Then some set covers 0. It then must also cover $1/n$, $n > m$ for some m. The rest of the points $1/n$, $n = 1, 2, \ldots, m$ can obviously be covered by a finite number of sets. These sets together with the set covering 0 form a finite subcover. So K is compact. □

(2) Problem 13 on page 44 of Rudin.

Solution. The set $U = \{1/m + 1/n | m, n \in \mathbf{N}\} \cup \{0\}$ suffices. Its limit points are $1/m$, $m \in \mathbf{N}$ and 0, which forms a countable set. To see that there are no additional limit points observe that if x is a limit point then there is an infinite sequence in U approaching x. Both m and n can not grow excessively large, and by symmetry we can assume m is bounded by some value and n grows large. As n grow very large (as must happen in our infinite sequence), $1/n$ grows so small that m must assume only a single value (or else $1/m + 1/n$) will differ significantly from x. Therefore sufficiently far into our sequence, m becomes fixed and n grows. Thus the sequence must converge to $1/m$, so $x = 1/m$ as desired. Since they are in U, U is closed. U is also bounded by 2. So by Heine-Borel, U is compact. □

(3) Problem 16 on page 44 of Rudin.

Solution. First of all, it suffices to consider $E = \{p | p \in (\sqrt{2}, \sqrt{3})\}$, as the negative case can be treated similarly.

E is clearly bounded by $\sqrt{2}$ and $\sqrt{3}$.

Consider a limit point p of E in $\mathbf{Q}$. It cannot be greater than $\sqrt{3}$ since we can then form a neighborhood around p which lies entirely outside the interval. Similarly, it cannot be less than $\sqrt{2}$. Since $p \in \mathbf{Q}$, it cannot be either of the endpoints of the interval. So it must be inside the interval and thus inside E. So E is closed.

E is open since $E = (\sqrt{2}, \sqrt{3}) \cup \mathbf{Q}$ and that is an open set in R.

In R, E is not closed since $\sqrt{2}$ is a limit point. So E is not compact in R. Since compactness is intrinsic, E is not compact in $\mathbf{Q}$. $\square$