Solution Set 10: Part A

Benjamin Bakker
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Prof. Berger

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A.1. *Find the rank of the matrix $A = (a_{ij})_{1\leq i,j\leq n}$ in the following cases:*

(1) $a_{ij} = i + j - 1$.

Solution. The image of a matrix is spanned by its columns (actually the vectors whose coordinates are a column). The general strategy for these problems will be to use column operations to make as many columns as we can 0, and then show that the remaining columns are linearly independent so we know we can't reduce it any further.

For $n \geq 2$, the matrix is

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ 2 & 3 & & n+1 \\ \vdots & \vdots & & \vdots \\ n & n+1 & \cdots & n+n-1 \end{pmatrix}$$

Multiplying the first by -1 and adding the second, we obtain $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ in the first column. We can then subtract the sum of the second column and $j-2$ times the first from the jth column to get all 0's in the jth column. Thus, we have

$$\begin{pmatrix} 1 & 2 & 0 & \cdots & 0 \\ 1 & 3 & 0 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & n+1 & 0 & \cdots & 0 \end{pmatrix}$$

The first two columns are clearly linearly independent, so the rank is 2. For $n = 1$, the rank is 1.

(2) $a_{ij} = \cos(\alpha(i + j))$ *where* $\alpha \in \mathbb{R}$.

Solution. For $n \geq 2$, the matrix is

$$\begin{pmatrix} \cos 2\alpha & \cos 3\alpha & \cdots & \cos(n+1)\alpha \\ \cos 3\alpha & \cos 4\alpha & & \cos(n+2)\alpha \\ \vdots & & & \\ \cos(n+1)\alpha & \cos(n+2)\alpha & & \cos 2n\alpha \end{pmatrix}$$

Note that $\cos(i+2)\alpha = \cos(i+1)\alpha\cos\alpha - \sin(i+1)\alpha\sin\alpha$, so if C_1 is the first column and C_2 the second, as long as α is not a multiple of π,

$$(-\sin\alpha)(\sin(i+1)\alpha)_{1\leq i\leq n} + (\cos\alpha)C_1 = C_2 \Rightarrow (\sin(i+1)\alpha)_{1\leq i\leq n} = \cot\alpha C_1 - \csc\alpha C_2$$

and therefore we may reduce to

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha & \cos 4\alpha & \cdots & \cos(n+1)\alpha \\ \cos 3\alpha & \sin 3\alpha & \cos 5\alpha & & \cos(n+2)\alpha \\ \vdots & \vdots & \vdots & & \vdots \\ \cos(n+1)\alpha & \sin(n+1)\alpha & \cos(n+3)\alpha & \cdots & \cos 2n\alpha \end{pmatrix}$$

Let C_2' be the new second column. We can make the jth column by $[\cos(j-1)\alpha]C_1 - [\sin(j-1)\alpha]C_2'$, and thus the image is spanned by the first two columns. For $n \geq 2$, these two are clearly independent for α not a multiple of π, for if

$$0 = a\begin{pmatrix}\cos 2\alpha\\ \cos 3\alpha\end{pmatrix} + b\begin{pmatrix}\sin 2\alpha\\ \sin 3\alpha\end{pmatrix}$$

for a, b not both zero, then

$$a\cos 3\alpha = a\cos\alpha\cos 2\alpha - a\sin\alpha\sin 2\alpha; \qquad b\sin 3\alpha = b\sin\alpha\cos 2\alpha + b\cos\alpha\sin 2\alpha$$

$$\Rightarrow 0 = \sin\alpha(b\cos 2\alpha - a\sin 2\alpha) \Rightarrow \begin{pmatrix}b & -a\\ a & b\end{pmatrix}\begin{pmatrix}\cos 2\alpha\\ \sin 2\alpha\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}\cos 2\alpha\\ \sin 2\alpha\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

since by assumption $\sin\alpha \neq 0$ and the above matrix is invertible as long as a, b are not both zero. We thus have a contradiction, and the rank is 2. For $\alpha = n\pi$ for some $n \in \mathbb{Z}$, the matrix is

$$\begin{pmatrix}1 & -1 & 1 & \cdots\\ -1 & 1 & -1 & \\ 1 & -1 & 1 & \\ \vdots & & & \end{pmatrix} \quad \text{or} \quad \begin{pmatrix}1 & 1 & 1 & \cdots\\ 1 & 1 & 1 & \\ 1 & 1 & 1 & \\ \vdots & & & \end{pmatrix}$$

depending on whether n is odd or even, and the rank is clearly 1. If $n = 1$, then the rank is 0 if α is an odd half-multiple of $\pi/2$ ($\frac{2n+1}{2}\frac{\pi}{2}$), and 1 otherwise.

(3) $a_{ij} = (\alpha + i + j)^2$ *where* $\alpha \in \mathbb{R}$.

Solution. The matrix is

$$\begin{pmatrix}(\alpha+2)^2 & (\alpha+3)^2 & \cdots & (\alpha+n+1)^2\\ (\alpha+3)^2 & (\alpha+4)^2 & & (\alpha+n+1)^2\\ \vdots & \vdots & & \vdots\\ (\alpha+n+1)^2 & (\alpha+n+2)^2 & \cdots & (\alpha+2n)^2\end{pmatrix}$$

The jth column has ith entry $(C_j)_i = (\alpha+i+j)^2 = (\alpha+i+j-1)^2 + 2(\alpha+i+j-1)+1$, and thus

$$C_j = C_{j-1} + 2\begin{pmatrix}1\\2\\ \vdots\\ n\end{pmatrix} + (2\alpha+2j-1)\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

and therefore we can build every column C_j from the three columns

$$C_1 = \begin{pmatrix}(\alpha+2)^2\\ (\alpha+3)^2\\ \vdots\\ (\alpha+n+1)^2\end{pmatrix}, \; C_2' = \begin{pmatrix}1\\2\\ \vdots\\ n\end{pmatrix}, \; C_3' = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

C_2', C_3' are in the image; subtracting C_2 from C_3 and C_1 from C_2 we obtain

$$\begin{pmatrix}(\alpha+2)^2 & 2(\alpha+3)+1 & 2(\alpha+4)+1 & \cdots & (\alpha+n+1)^2\\ (\alpha+3)^2 & 2(\alpha+4)+1 & 2(\alpha+5) & & (\alpha+n+1)^2\\ \vdots & \vdots & \vdots & & \vdots\\ (\alpha+n+1)^2 & 2(\alpha+n+1) & 2(\alpha+n+2)+1 & \cdots & (\alpha+2n)^2\end{pmatrix}$$

Subtracting the second from the third we then obtain C_3', and we can then use that to make the second column C_2'.

Moreover, these three are linearly independent for $n \geq 3$. If $aC_1 + bC_2' + cC_3' = 0$, then all n rows of $aC_1 + bC_2'$ are equal. Thus, $a(\alpha+2)^2 + b = a(\alpha+3)^2 + 2b = a(\alpha+4)^2 + 3b$; the first two give $2a\alpha + 5a + b = 0$, and the second and third give $2a\alpha + 7a + b = 0$, so $a = 0$ and then $b = c = 0$.

For $n = 2$, the first two are linearly independent, so the rank is 2 by a similar argument, and for $n = 1$ the rank is 1 if $\alpha \neq -2$, and 0 otherwise.

A.2. *Let E be a vector space over a field K, and let F be a subspace (not equal to E). Prove that* $\mathrm{Vect}(E \setminus F) = E$ *(i.e. that every vector in E is a linear combination of vectors in $E \setminus F$). Make sure your proof works for every field K.*

Solution. $F \neq E$, so there is some nonzero $x \in E \setminus F$. For any $v \in F$, $v - x$ is in $E \setminus F$, for if $v - x \in F$ then since $v \in F$ we would have $v - (v - x) = x \in F$. We then have $v = (v - x) + x$.

A.3. *Let K be an infinite field, let V be a K-vector space of dimension n and let $W_1, \ldots, W_k$ be subspaces of V of dimension $\leq n - 1$. Prove that $V \neq \cup_{i=1}^{k} W_i$.*

Solution. We will first show that for any n-dimensional K-vector space V, $n \geq 2$, there are infinitely many different $n - 1$ dimensional subspaces. Take $v_1, \ldots, v_n$ to be a basis of V, and let $W_\lambda = \mathrm{Vect}(v_1 + \lambda v_n, v_2, \ldots, v_{n-1})$ for $\lambda \in K$. Each W_λ has dimension $n - 1$, and if $W_\lambda = W_{\lambda'}$, then for some $a_i \in K$

$$v_1 + \lambda' v_n = a_1(v_1 + \lambda v_n) + a_2 v_2 + \cdots + a_{n-1} v_{n-1}$$

$$\Rightarrow 0 = (a_1 - 1)v_1 + a_2 v_2 + \cdots + a_{n-1} v_{n-1} + (a_1 \lambda - \lambda')$$

and since $(v_i)_{1 \leq i \leq n}$ is a basis, we must have $a_1 = 1$ and thus $\lambda = \lambda' \Rightarrow W_\lambda = W_{\lambda'}$. If K is infinite, we then have distinct $n - 1$ dimensional subspace W_λ for every $\lambda \in K$.

Now for the claim in the problem; our proof will be by induction on n. The $n = 1$ case is trivial since the W_i are all (0). In the general case, assume we have the claim for dimension $\leq n - 1$ spaces, $n \geq 2$. By the above, there are infinitely many $n - 1$ dimensional subspaces of V; pick W to be different from all of the W_i. Then $W_i \cap W \neq W$ for all i, and so each $W_i \cap W$ is a subspace of W of strictly lower dimension. Thus, there is some $v \in W$ which is not contained in any $W_i \cap W$, and thus $v \in V$ is not contained in any W_i.

Show that this is not always true if K is finite.

Solution. If $\mathrm{Card}(K) = p$, then an n dimensional K-vector space has p^n elements. If we enumerate the vectors w_i and take $W_i = \mathrm{Vect}(w_i)$, then $V = \cup_i W_i$.

A.4. *Let V be a K-vector space of dimension N and let W_1 and W_2 be two subspaces of the same dimension $r \leq n - 1$. Prove that there exists a subspace U of V such that $V = W_1 \oplus U$ and $V = W_2 \oplus U$.*

Solution. First note that (A) if we have two spaces $W_1, W_2 \subset V$ of dimension strictly less than $\dim V$, then $V \neq W_1 \cap W_2$, or else $\mathrm{Vect}(V \setminus W_1) = \mathrm{Vect}(W_2) = W_2 \neq V$, contrary to A.2.

By induction on $n - r$. The $n - r = 1$ case is trivial by (A), for there is some (nonzero) $v \in V$ not in either W_i, so take $U = \mathrm{Vect}(v)$. For the general case take $\dim W_i = n - k$ and assume the claim for $n - r < k$. Again by (A) there exists a vector $v \in V$ that is neither in W_i. If we take $W_i' = \mathrm{Vect}(W_i, v)$ for $i = 1, 2$, W_i' are two $r + 1$ dimensional subspaces of V, so applying the induction hypothesis for $n - r = k - 1$ yields a subspace $U' \subset V$ with $V = W_i' \oplus U'$. Let $U = \mathrm{Vect}(U', v)$. Then $U \cap W_i = (0)$ because $U' \cap W_i' = (0)$ and $v \notin W_i$. Since $V = W_i' \oplus U'$, W_i and U span V, so $V = W_i \oplus U$.

A.5. *Let V be a finite dimensional K-vector space and let $u, v, w \in \mathcal{L}(V, V)$ be three endomorphisms of V. Prove the following inequalities.*

(1) $\mathrm{rk}(u + v) \leq \mathrm{rk}(u) + \mathrm{rk}(v)$

Solution. Any vector $x \in \mathrm{im}(u + v)$ is in $\mathrm{im}(u) + \mathrm{im}(v)$, for we must have $x = (u + v)(y) = u(y) + v(y) \in \mathrm{im}(u) + \mathrm{im}(v)$ for some $y \in V$. Thus, $\mathrm{im}(u + v) \subset \mathrm{im}(u) + \mathrm{im}(v)$, and the above inequality then follows.

(2) $\mathrm{rk}(u + v) \geq |\mathrm{rk}(u) - \mathrm{rk}(v)|$

Solution. By (1), $\mathrm{rk}(v) = \mathrm{rk}(u-(u+v)) \leq \mathrm{rk}(u+v)+\mathrm{rk}(u)$ since obviously $\mathrm{rk}(-v) = \mathrm{rk}(v)$. Thus, $\mathrm{rk}(u+v) \geq \mathrm{rk}(v)-\mathrm{rk}(u)$. Interchanging u and v we obtain $\mathrm{rk}(u+v) \geq \mathrm{rk}(u)-\mathrm{rk}(v)$, and the above inequality follows.

(3) $\inf(\mathrm{rk}(u), \mathrm{rk}(v)) \geq \mathrm{rk}(uv) \geq \mathrm{rk}(u) + \mathrm{rk}(v) - \dim(V)$

Solution. Let the dimension of V be n. The first inequality is easy, since $\mathrm{im}(uv) \subset \mathrm{im}(u)$ and $\ker(v) \subset \ker(uv)$ implies

$$\dim \mathrm{im}(uv) = n - \dim \ker(uv) \leq n - \dim \ker(v) = \dim \mathrm{im}(v)$$

For the second inequality, let $A = \mathrm{im}(v) \subset V$ and consider the restriction $u|_A : \mathrm{im}(v) \to V$. Of course, $\mathrm{im}(u|_A) = \mathrm{im}(uv)$, but $\ker(u|_A) = A \cap \ker(u)$, and thus $\dim \ker(u|_A) \leq \dim \ker(u)$. This implies that

$$\mathrm{rk}(uv) = \mathrm{rk}(u|_A) = \dim A - \dim \ker(u|_A) \geq \dim \mathrm{im}(v) - \dim \ker(u) = \mathrm{rk}(v) + \mathrm{rk}(u) - \dim(V)$$

(4) $\mathrm{rk}(uv) + \mathrm{rk}(vw) \leq \mathrm{rk}(v) + \mathrm{rk}(uvw)$

Solution. Note that v maps $\ker(uv)$ onto $\ker(u) \cap \mathrm{im}(v)$; it is clear that $v(\ker(uv)) \subset \ker(u) \cap \mathrm{im}(v)$, and if $x \in \ker(u) \cap \mathrm{im}(v)$, $x = v(y)$ for some $y \in V$ and $v(y) \in \ker(u) \Rightarrow y \in \ker(uv) \Rightarrow x \in v(\ker(uv))$. The kernel of this map is, of course, $\ker(v)$ (since $\ker(v) \subset \ker(uv)$), so $\dim \ker(uv) = \dim \ker(v) + \dim \ker(u) \cap \mathrm{im}(v)$. It then follows by $n = \dim \ker(u) + \mathrm{rk}(u)$ that

$$\mathrm{rk}(v) - \mathrm{rk}(uv) = \dim \ker(u) \cap \mathrm{im}(v)$$

and similarly

$$\mathrm{rk}(vw) - \mathrm{rk}(uvw) = \dim \ker(u) \cap \mathrm{im}(vw)$$

Since $\mathrm{im}(v) \supset \mathrm{im}(vw)$, the first is greater than or equal to the second, thus

$$\mathrm{rk}(v) - \mathrm{rk}(uv) \geq \mathrm{rk}(vw) - \mathrm{rk}(uvw) \Rightarrow \mathrm{rk}(v) + \mathrm{rk}(uvw) \geq \mathrm{rk}(uv) + \mathrm{rk}(vw)$$

Use the last inequality to prove that if $k \geq 1$, and u^k denote $u \times \cdots \times u$ (k times) then:

$$\mathrm{rk}(u^k) \leq \frac{1}{2}\left(\mathrm{rk}(u^{k+1}) + \mathrm{rk}(u^{k-1})\right)$$

Solution. Set $v = u^{k-1}$, $u = w$ and apply (4); we get

$$2\mathrm{rk}(u^k) \leq \mathrm{rk}(u^{k-1} + \mathrm{rk}(u^{k+1})) \Rightarrow \mathrm{rk}(u^k) \leq \frac{1}{2}\left(\mathrm{rk}(u^{k+1}) + \mathrm{rk}(u^{k-1})\right)$$