Problem Set #8 Part B
Official Solutions
Total Points: 40

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(5) B1. Let H_g be the cyclic subgroup of G generated by $g \in G$. Then define the map $f : G \to \{H_g : g \in G\}$ which maps g to H_g. First suppose that for some $g \in G$ H_g is infinite. Then $H_g \cong \mathbf{Z}$, which has infinitely many subgroups. Thus we see that H has infinitely many subgroups, so G must also have infinitely many subgroups. Next, suppose that G does not have any elements with infinite order; then all H_g will be finite. Note that at most $|H_g|$ elements map to every H_g; thus if there were only finitely many H_g G would be finite, a contradiction. So there must be infinitely many H_g so G has infinitely many subgroups.

Note that G could have many more subgroups than just these cyclic subgroups.

(5) B2. First note that the number of left cosets of a group H is the same as the number of right cosets. Indeed, we see that if $g \in aH$, so $g = ah$ for some $h \in H$, then $g^{-1} = h^{-1}a^{-1} \in Ha^{-1}$. Thus there is a bijection between the left cosets and the right cosets, so the number of left cosets must be equal to the number of right cosets.

Now suppose that H is a subgroup of G with index 2, and let $g \in G$ be such that $g \notin H$. Then we know that there are two left cosets: H, gH, and two right cosets, $H, Hg^{-1} = Hg$ since $g \notin H$. Since $H = H$, we see that $gH = Hg$ so H is a normal subgroup.

(15) B3.

(1) (1) Suppose that $x, y \in Z$. We need to show that both xy and x^{-1} are in Z. First note that since $xg = gx$ we know that $gx^{-1} = x^{-1}g$ (by multiplying on the right and left by x^{-1}) so we see that if x commutes with all g then so does x^{-1}. Similarly, if $xg = gx$ and $yg = gy$ we have that $gxy = xgy = xyg$ so xy commutes with g. Thus if $x, y \in Z$ then $x^{-1}, xy \in Z$, so Z is a subgroup of G.

(4) (2) Suppose that G/Z is cyclic. That means that there exists $a \in G$ such that all cosets are of the form $a^m Z$ for some $m \in \mathbf{Z}$. Thus every element in G is of the

form $a^m z$ for some $m \in \mathbf{Z}$, $z \in Z$. Fix any $x, y \in G$ and write $x = a^m z_1$ and $y = a^n z_2$. Since the z_i commute with everything we have

$$xy = (a^m z_1)(a^n z_2) = a^m a^n z_1 z_2 = a^n a^m z_2 z_1 = (a^n z_2)(a^m z_1) = yx$$

so G is abelian.

(5) (3) Consider a group A, and suppose that $\#A = p$ a prime. Then we will show that A is cyclic. Suppose $x \in A$ is not the identity. Then the cyclic group generated by x is nontrivial; thus its order must be p and it must be the whole group. Therefore if the order of A is prime then A is cyclic.

Now consider G/Z. If $[G : H]$ is prime then G/Z is cyclic; thus if G is nonabelian $[G : Z]$ is not prime and is larger than 1, so $[G : Z] \geq 4$.

(5) (4) Fix $y \in G$. Note that $C_y = \{x \in G : xy = yx\}$ is a subgroup of G. Thus $\#C_y | \#G$, so if $\#C_y \neq \#G$, $\#C_y \geq \frac{1}{2}\#G$. If $\#C_y = \#G$ then $y \in Z$. Thus

$$\#\{(x, y) : xy = yx\} \leq \#Z\#G + \frac{1}{2}(\#G - \#Z)\#G = \frac{1}{2}\#Z\#G + \frac{1}{2}(\#G)^2.$$

However, since G is not abelian $\#Z \leq \frac{1}{4}\#G$. So

$$\#\{(x, y) : xy = yx\} \leq \frac{1}{8}(\#G)^2 + \frac{1}{2}(\#G)^2 = \frac{5}{8}(\#G)^2.$$

(10) B4. First note that S_4 has a subgroup of order 12. First note that S_4 contains only elements of orders $1, 2, 3, 4$; more exactly, it contains 1 element of order 1, 9 of order 2, 8 of order 3 and 6 of order 4. Consider the subset of S_4 that contains all bijections that fix exactly one element, the identity, and the bijections that switch two disjoint pairs of elements (of which there are 3). This set will contain exactly 12 elements. It also clearly contains the inverses of each of these bijections. Thus to show that it is a subgroup we simply need to show that it contains the compositions of any two of these, which is simple to check (so we will omit it here). We will call this group A_4.

Now we will show that A_4 has no subgroup of order 6. First note that, in a group of order 6 with no element of order 6 (since S_4 has no element of order 6, A_4 will not have one either) there will be elements only of orders $1, 2, 3$. Note, in addition, that there cannot be more than two elements of order 3. Since elements of order three come in pairs (the element and its inverse) if there are more than two elements of order 3 there will be four such elements. Thus there will only be one element of order 2. Call that element a, and let b be another non-identity element. Since $ab \neq e$ we know that $ab = c$ where c also has order 3. Also, by process of elimination we see that $bc = a$. However, then $e = a^2 = abc = c^2 \neq e$. Contradiction. Thus there cannot be more than two elements of order 3; thus there must be 3 elements of order two. However, since A_4 contains only three elements of order 2, we see that thus subgroup must contain all three of these elements. The product of any two of these elements will be another element of this form; thus the elements of order 2 (plus the identity) will form a subgroup of order 4 in a group of order 6. Contradiction. Thus A_4 has no subgroup of order 6.

So now we simply need to show that there will be no other subgroup of order 12. Suppose that we have two subgroups of order 12; call these A_4 and B_4. These will each have one coset. Let $X = A_4 \cap B_4$, which will clearly be a subgroup of A_4 and B_4. Let $a \in A_4 \backslash X$. Then we know that $aB_4 \cup B_4 = G$. In addition, if for $b \in B$ $ab \in A_4$, we see that $b \in A_4$ so $b \in X$. Thus $\#X = \#(A_4 \backslash X)$, so $\#X = 6$. Thus A_4 will have a subgroup of order 6 if there is another group; this is not the case. Thus there is no other subgroup of order 12, and we are done.

(5) B5. First note that if $g \in aH$ (for some $a \in G$) then $gHg^{-1} = aHa^{-1}$. Since $g \in aH$, we know that $g = ah$ for $h \in H$. Thus

$$gHg^{-1} = (ah)H(ah)^{-1} = ahHh^{-1}a^{-1} = aHa^{-1}.$$

Let A be the set of all $g \in G$ such that if $g_1, g_2 \in A$ then $g_1Hg_1^{-1} \neq g_2Hg_2^{-1}$, so $\cup_{g \in G} gHg^{-1} = \cup_{a \in A} aHa^{-1}$. We showed above that $\#A \leq [G : H]$. However, note that each aHa^{-1} will contain e. Thus

$$\#(\cup_{a \in A} aHa^{-1}) \leq 1 + (\#A)(\#H - 1) \leq 1 + [G : H](\#H - 1) = \#G - [G : H] + 1.$$

This will equal \$$G$ if and only if $H = G$. However, this is not the case, so we know that this will be strictly less than $\#G$, so it cannot be all of G.