Problem Set #7 Part B
Official Solutions
Total Points: 20 (+6)

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(10) B1. We will first find the limit of $(n+1)\int_0^1 x^n f(x)\,dx$. Since the ratio between the function that we want and this goes to 1 its limit will be the same as the limit of our function.

Let $M = \sup_{[0,1]}|f|$. Fix $\epsilon > 0$. Since f is uniformly continuous we know that there exists $\delta > 0$ such that if $|x - x'| < \delta$ then $|f(x) - f(x')| < \epsilon$. Consider the two integrals $\int_0^{1-\delta} x^n f(x)\,dx$ and $\int_{1-\delta}^1 x^n f(x)\,dx$. Then

$$(n+1)\left|\int_0^{1-\delta} x^n f(x)\,dx\right| \le (n+1)\int_0^{1-\delta} |x^n||f(x)|\,dx \le (n+1)\int_0^{1-\delta} x^n M\,dx = M(1-\delta)^{n+1}.$$

On the other hand,

$$(n+1)\int_{1-\delta}^1 x^n f(x)\,dx \ge (n+1)\int_{1-\delta}^1 x^n (f(1)-\epsilon)\,dx = (f(1)-\epsilon) - (1-\delta)^{n+1}(f(1)-\epsilon).$$

As $n \to \infty$ the first of these integrals goes to 0, and the second integral goes to $f(1) - \epsilon$. Thus we see that the limit of the total integral as $n \to \infty$ is at least $f(1)$. On the other hand,

$$(n+1)\int_{1-\delta}^1 x^n f(x)\,dx \le (n+1)\int_{1-\delta}^1 x^n (f(x)+\epsilon) = (f(1)+\epsilon) - (1-\delta)^{n+1}(f(1)+\epsilon).$$

Thus the second integral is also less than or equal to $f(1)$. So the limit of the integral is going to be $f(1)$.

Now we want to write I_n as $f(1) + g(n)$ where $g(n) = o(1)$. Note

$$\int_0^1 nx^n f(x) = \frac{n}{n+1}\left(f(x)x^{n+1}\Big|_0^1 - \int_0^1 x^{n+1} f(x)\,dx\right) = \frac{n}{n+1}\left(f(1) - \int_0^1 x^{n+1} f'(x)\,dx\right).$$

Then

$$n\left(n\int_0^1 x^n f(x)\,dx - f(1)\right) = \frac{n^2}{n+1}f(1) - \frac{n^2}{n+1}\int_0^1 x^{n+1}f'(x)\,dx - nf(1).$$

Let $C(n) = n\int_0^1 x^{n+1} f'(x)\,dx$. We know that $\lim_{n\to\infty} C(n) = f'(1)$. Then the above simplifies to

$$-\frac{n}{n+1}(C(n) + f(1))$$

which goes to $-f(1) - f'(1)$ as $n \to \infty$. Thus we see that $\lim_{n\to\infty} n(I_n - f(1))$ exists, so

$$I_n \sim f(1) - \frac{1}{n}(f(1) + f'(1)) + o(1/n).$$

(10 (+6)) B2.

(5) (1) Note that the integrand is a C^∞ function of z which is integrated on a compact set. Thus, by a simple induction on m, we know that

$$\frac{d^m}{dz^m} J_n(z) = \frac{1}{\pi}\frac{d^m}{dz^m}\int_0^\pi \cos(n\theta - z\sin\theta)\,d\theta = \frac{1}{\pi}\int_0^\pi \frac{\partial^m}{\partial z^m}\cos(n\theta - z\sin\theta)\,d\theta$$

which will exist and be continuous for all m.

(5) (2) Note that

$$\begin{aligned}
J_{n-1}(z) - J_{n+1}(z) &= \frac{1}{\pi}\int_0^\pi \cos((n-1)\theta - z\sin\theta)\,d\theta - \frac{1}{\pi}\int_0^\pi \cos((n+1)\theta - z\sin\theta)\,d\theta \\
&= \frac{1}{\pi}\int_0^\pi (\cos((n-1)\theta - z\sin\theta) - \cos((n+1)\theta - z\sin\theta))\,d\theta \\
&= \frac{1}{\pi}\int_0^\pi 2\sin(n\theta - z\sin\theta)\sin\theta\,d\theta.
\end{aligned}$$

Also

$$J_n'(z) = \frac{1}{\pi}\int_0^\pi \frac{\partial}{\partial z}\cos(n\theta - z\sin\theta)\,d\theta = \frac{1}{\pi}\int_0^\pi \sin(n\theta - z\sin\theta)\sin\theta\,d\theta.$$

Thus we see that

$$J_{n-1}(z) - J_{n+1}(z) = 2J_n'(z).$$

Note that for $n = 0$ the first formula doesn't work. However, note that

$$\begin{aligned}
J_1(z) &= \frac{1}{\pi}\int_0^\pi \cos(\theta - z\sin\theta)\,d\theta \\
&= \frac{1}{\pi}\int_0^\pi \cos\theta\cos(z\sin\theta) + \frac{1}{\pi}\int_0^\pi \sin\theta + \sin(z\sin\theta)\,d\theta \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(\theta - \pi/2)\cos(z\sin(\theta - \pi/2))\,d\theta - J_0'(z) \\
&= -\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin\theta\cos(z\cos\theta) - J_0'(z) \\
&= -J_0'(z)
\end{aligned}$$

because the first integral is a symmetric integral of an odd function.