Problem Set #6 Part B
Official Solutions
Total Points: 48

Inna Zakharevich

November 4, 2003

(8) B1.

(5) (1) Note that, by Taylor's expansion, we have that

$$\log(1+x) = x - x^2 \int_0^1 \frac{(1-u)}{(1+ux)^2}\,du.$$

From this, we see that

$$x - \log(1+x) = x^2 \int_0^1 \frac{(1-u)}{(1+ux)^2}\,dx \le x^2 \int_0^1 (1-u)\,du = \frac{1}{2}x^2.$$

Thus if we set $M = 1/2$ it will satisfy the desired equation. From the expansion it is also clear that $x - \log(1+x) \ge 0$ for all $0 \le x \le 1$.

(3) (2) Then

$$\frac{1}{k} - \log\frac{k+1}{k} \le \frac{1}{2k^2}$$

so the infinite sum of these will converge. However, note that

$$\sum_{k=1}^{n}\left(\frac{1}{k} - \log\frac{k+1}{k}\right) = \sum_{k=1}^{n}\frac{1}{k} - \log(n+1) = \left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) + \log\frac{n}{n+1}.$$

We know that this converges. In addition, we know that $\log n/(n+1)$ converges to 0. Thus the limit of this will be the same as the limit of $\sum 1/k - \log n$, so the limit of the desired sum exists.

(8) B2. We know that $\int_a^b (f - tg)^2 \ge 0$. Expanding this, we see that $t^2\int_a^b g^2 - 2t\int_a^b fg + \int_a^b f^2 \ge 0$. Thus we have a quadratic polynomial in t which is always ≥ 0, so its discriminant is ≤ 0. This translates into $(2\int_a^b fg)^2 - 4(\int_a^b f^2)(\int_a^b g^2) \le 0$ which is the required inequality. Alternatively, we could plug $t = (\int_a^b fg)/(\int_a^b g^2)$ into the polynomial to get the inequality.

(24) B3.

(8) (1) We will first show this for two step functions. We will use the result of A2, where we will spproximate the integral of a ruled f with functions that are equal to f at every rational point with denominator n. Then we need to show that

$$\sum_{i=1}^{n} a_i b_i x_i \ge \left(\sum_{i=1}^{n}\frac{a_i}{n}\right)\left(\sum_{i=1}^{n}\frac{b_i}{n}\right).$$

First note that, since the sequence is nonecreasing, we have

$$(a_i - a_j)(b_i - b_j) \geq 0 \Longrightarrow a_i b_i + a_j b_j \geq a_i b_j + a_j b_i.$$

However,

$$\begin{aligned}
\left(\sum_{i=1}^{n} \frac{a_i}{n}\right)\left(\sum_{i=1}^{n} \frac{b_i}{n}\right) &= \sum_{k=0}^{n-1}\sum_{i=1}^{n} \frac{a_i}{n}\frac{b_{i+k}}{n} \\
&\leq n \sum_{i=1}^{n} \frac{a_i b_i}{n^2} \\
&= \sum_{i=1}^{n} \frac{a_i b_i}{n}
\end{aligned}$$

as desired.

Now, since this is true for any n, it will be true in the limit as $n \to \infty$. Thus this is also true in the continuous formulation.

(4) (2) We will use Cauchy-Schwartz. First note that f does not cross 0; thus it is always either greater than 0 or less than 0. Without loss of generality we will assume that f is always greater than 0. Then let $g(x) = \sqrt{f(x)}$ and let $h(x) = \sqrt{1/f(x)}$. Then we know that

$$\int_0^1 f \int_0^1 1/f \geq \left(\int_0^1 gh\right)^2 = 1$$

as desired.

(6) (3) Let $M = |f'|_{[0,1]}$. We know that $|f'(x)| \leq M$. Thus $-M \leq f'(x) \leq M$, so $|f(x)| \leq Mx$. Similarly, we see that $|f'(1-x)| \leq M$, so $|f(x)| \leq M(1-x)$. Let $h(x)$ be Mx if $0 \leq x \leq 1/2$ and $M(1-x)$ if $1/2 \leq x \leq 1$, so $|f| \leq h$. Then

$$\left|\int_0^1 f\right| \leq \int_0^1 |f| \leq \int_0^1 h = \frac{1}{4}M.$$

Note that we can't replace $1/4$ with a smaller number, because we can approximate the function h with a $\mathcal{C}^1$ function by replacing the cusp at $1/2$ by a tiny parabola so that the derivatives line up.

(6) (4) We will do this by integration by parts. Note that $(f(t)^2)' = 2f(t)f'(t)$ and that $(\cot t)' = -\csc^2 t = -(1+\cot^2 t)$. Thus

$$\int_0^\pi f(t)f'(t)\cot t\, dt = \frac{1}{2}f(t)^2 \cot t\Big|_0^\pi - \int_0^\pi \frac{1}{2}f(t)^2(-(1+\cot^2 t))\, dt = \frac{1}{2}\int_0^\pi f(t)^2(1+\cot^2 t)\, dt$$

because $f(0) = f(\pi) = 0$.

Now note that $\int_0^\pi (f(t)\cot t - f'(t))^2\, dt \geq 0$. Thus we see that $\int_0^\pi f'(t)\, dt \geq 2\int_0^\pi f(t)f'(t)\cot t\, dt - f^2\cot(t)\, dt$. Then

$$\begin{aligned}
\frac{1}{2}\int_0^\pi f(t)^2 &= \int_0^\pi f(t)^2 \csc^2 t\, dt - \int_0^\pi f(t)^2 \cot^2 t\, dt \\
&= 2\int_0^\pi f(t)f'(t)\cot t\, dt - \int_0^\pi f(t)^2\cot^2 t\, dt \\
&\leq \int_0^\pi f'(t)^2.
\end{aligned}$$

(8) B4. We can expand $\sin\sinh(x)$ and $\sinh\sin x$ in their Taylor expansion. We know that $\sin x = x - x^3/3! + x^5/5! - x^7/7! + O(x^9)$ and $\sinh x = x + x^3/3! + x^5/5! + x^7/7! + O(x^9)$. So we can expand as

$$\begin{aligned}
\sinh\sin x &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + O(x^9)\right) + \frac{1}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)\right)^3 + \frac{1}{5!}\left(x - \frac{x^3}{3!} + O(x^5)\right)^5 \\
&\quad + \frac{1}{7!}(x + O(x^3))^7 \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + O(x^9)\right) + \frac{1}{3!}\left(x^3 - \frac{x^5}{2} + \frac{13x^7}{120} + O(x^9)\right) \\
&\quad + \frac{1}{5!}\left(x^5 - \frac{5x^7}{6} + O(x^9)\right) + \frac{1}{7!}(x^7 + O(x^9) \\
&= x - \frac{1}{15}x^5 + \frac{1}{90}x^7 + O(x^9). \\
\sin\sinh x &= \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + O(x^9)\right) - \frac{1}{3!}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)\right)^3 + \frac{1}{5!}\left(x + \frac{x^3}{3!} + O(x^5)\right)^5 \\
&\quad - \frac{1}{7!}(x + O(x^3))^7 \\
&= \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + O(x^9)\right) - \frac{1}{3!}\left(x^3 + \frac{x^5}{2} + \frac{13x^7}{120} + O(x^9)\right) \\
&\quad + \frac{1}{5!}\left(x^5 + \frac{5x^7}{6} + O(x^9)\right) - \frac{1}{7!}(x^7 + O(x^9) \\
&= x - \frac{1}{15}x^5 - \frac{1}{90}x^7 + O(x^9).
\end{aligned}$$

Note that the coefficients of x^7 are different. Thus the Taylor expansion for $\sin\sinh x - \sinh\sin x = -1/45x^7 + O(x^9)$. Since $-1/45 = f^{(7)}(0)/7!$, we see that $f^{(7)}(0) = -112$, and is the first nonzero derivative.