Solution Set 6: Part A

Benjamin Bakker
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Prof. Berger

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A.1. Filling holes. *Let f, g be two continuous functions on the interval $[a, b]$ and fix $c \in (a, b)$. Suppose that f is C^1 on $[a, b] \setminus \{c\}$ and that $f' = g$ on $[a, b] \setminus \{c\}$.*

(1) *Prove that f is differentiable at c and that $f' = g$ on $[a, b]$.*

Solution. Consider the function $\alpha(x) = f(x) - \int_c^x g$. By the fundamental theorem of calculus, $G(x) = \int_c^x g$ is C^1 on $[a, b]$, and therefore α is C^1 on $[a, b] \setminus \{c\}$; note that $\alpha(c) = f(c)$ and $\alpha' = 0$ on $[a, b] \setminus \{c\}$. It will follow that f is differentiable at c and $f'(c) = g(c)$ once we prove that α is differentiable at c with $\alpha'(c) = 0$ since $G'(c) = g(c)$. In that case, $f' = g$ on $[a, b]$, and as g is continuous, f is C^1 on $[a, b]$.

For any $h > 0$, by the mean value theorem for integrals, there exists $y \in (c, c + h)$ such that $\int_c^{c+h} g = hg(y)$; since f is continuous on $[a, b]$ and differentiable on $(c, c + h)$, the mean value theorem for derivatives guarantees a $z \in (c, c + h)$ such that $hf'(z) = hg(z) = f(c + h) - f(c)$. Thus,

$$\alpha(c + h) = f(c + h) - \int_c^{c+h} g = f(c) + h(g(z) - g(y))$$

As $h \longrightarrow 0$, $z, y \longrightarrow c$ and thus $g(z) - g(y) \longrightarrow 0$ since g is continuous on $[a, b]$. By definition of the derivative, $\alpha'(c) = 0$. □

(2) *Prove that the function f defined by $f(0) = 1$ and $f(x) = x/\sin x$ is C^1 on $[-\pi/2, \pi/2]$.*

Solution. Let $f = x/\sin x$ on $[-\pi/2, \pi/2] \setminus \{0\}$. x and $\sin x$ are both nonzero, C^1, and with nonzero derivative on $(-\pi/2, \pi/2) \setminus \{0\}$, so f is C^1 on $[-\pi/2, \pi/2] \setminus \{0\}$ (the way we defined it, f is C^1 on $[a, b]$ if it has a continuous derivative on (a, b)). Let $f(0) = 1$. By L'Hôpital's rule, $\lim_{x\to 0} f(x) = \lim_{x\to 0} 1/\cos x = 1 = f(0)$, and thus f is continuous on $[a, b]$. By part (1), f is C^1 on $[-\pi/2, \pi/2]$. □

A.2 Some Riemann sums I. Let $I = [0, 1]$ and let $f : I \to \mathbb{R}$ be a continuous function. For $n \geq 1$, define a step function f_n by:

$$f_n(x) = \begin{cases} f(1/n) & \text{if } 0 \leq x \leq 1/n \\ f(k/n) & \text{if } 0 \leq k \leq n \text{ and } (k-1)/n < x \leq k/n \end{cases}$$

(1) *Prove that $|f_n - f|_I \to 0$ as $n \to +\infty$.*

Solution. f is continuous, and therefore uniformly continuous, so for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Take N such that $\frac{1}{N} < \delta/2$. For any $n > N$ and $x \in I$, there is a k for which $x \in B(k/n, 1/n)$, in which case $f_n(x) = f(k/n)$ or $f_n(x) = f((k + 1)/n)$; by uniform continuity, since $x, k/n, (k + 1)/n$ are all in $B(k/n, \delta)$, their images under f are no more than ϵ away from each other, and therefore $|f(x) - f_n(x)| < \epsilon \; \forall x \in I \Rightarrow \sup_{y\in I} |f(y) - f_n(y)| \leq \epsilon$. Thus, $|f - f_n|_I \longrightarrow 0$. □

(2) *Use this to show that*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f \qquad (1)$$

Solution. $\int_a^b$ is a continuous function from the set $\mathcal{R}$ of ruled functions on $I = [a, b]$ (endowed with the uniform metric $|\cdot|_I$) into $\mathbb{R}$, so for any convergent sequence $f_n \in \mathcal{R}$ (with respect to $|\cdot|_I$, i.e., a uniformly convergent sequence of functions) with limit $f \in \mathcal{R}$, the sequence $\int_0^1 f_n$ converges to $\int_0^1 f$ in $\mathbb{R}$. f_n is a step function, so by definition,

$$\int_0^1 f_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

and equation (1) follows. □

(3) *Prove that the above formula still holds if f is a ruled function on I.*

Solution. If f is ruled, then by definition there is a sequence g_n of step functions uniformly converging to f. Note that f is bounded (problem 3.1 on the midterm), as are the g_n, so there is some B such that $B > |f(x)|$ and $B > |g_n(x)|$ for all n.

Choose $\epsilon > 0$. There exists N such that $|f - g_N|_I < \epsilon/3$. Let $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be the partition of I associated with g_N, so g_N is a constant g_i on the interval (x_i, x_{i+1}) for $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. For every $x \in (x_i, x_{i+1})$, we have $|f(x) - g_i| < \epsilon$. In particular, for any rational $k/m \in (x_i, x_{i+1})$, $|f(k/m) - g_i| < \epsilon$ and therefore $|f_m(x) - g_N(x)| < \epsilon$ for all $x \in (x_i, k_i/m)$, where k_i is the largest integer such that $k_i/m < x_{i+1}$; let A_{nm} be the union of all the $(x_i, k_i/m)$. Thus, f_m is close to g_n on A_{nm}, and we need only show that the possible differences between f_m and g_n at points x not in the closure of A_{mn}[1], located to the left of some x_i with no k/m between x and x_i do not contribute to the integral $\int_0^1 f_m$ in the limit $m \longrightarrow \infty$.

Take $M > 6BN/\epsilon$ so that there is a rational k/m no more than $\frac{\epsilon}{6BN}$ to the left of every x_i ($i > 0$) for any $m > M$. By the above, $\int_0^1 f_m - \int_0^1 g_n$ splits into two pieces—on A_{nm}, f_m and g_n differ by no more than $\epsilon/3$, while on $I \setminus A_{nm}$, whose total length is $N \cdot \frac{\epsilon}{6BN} = \frac{\epsilon}{6B}$, they differ by at most $2B$, so

$$\left|\int_0^1 f_m - \int_0^1 g_N\right| \leq \frac{\epsilon}{3} + 2B \cdot \frac{\epsilon}{6B} = \frac{2\epsilon}{3} \qquad \forall m > M$$

and since $\left|\int_0^1 g_N - \int_0^1 f\right| < \epsilon/3$ by construction,

$$\left|\int_0^1 f_m - \int_0^1 f\right| < \epsilon \qquad \forall m > M$$

It then follows that

$$\int_0^1 f = \lim_{m\to\infty}\int_0^1 f_m = \lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m} f(k/m)$$

as claimed.

Note: f_n does **not** necessarily uniformly converge to f if f is ruled; what the above shows is that the set of points at which f and f_n always have a finite difference is small enough that in the limit $m \longrightarrow \infty$, it doesn't matter. □

A.3 Some Riemann sums II. *Using the results of the previous exercise, compute the following limits:*

$$\text{(a)} \quad \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{k+n}, \qquad \text{(b)} \quad \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{n}{k^2+n^2}, \qquad \text{(c)} \quad \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{k^3}{n^4}\cos\left(\frac{4k}{n}\right)$$

Solution. We can rewrite the sums as

$$\text{(a)} \quad \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k/n+1}, \qquad \text{(b)} \quad \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{(k/n)^2+1}, \qquad \text{(c)} \quad \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3\cos\left(\frac{4k}{n}\right)$$

[1]We don't really care about the value of g_N at the endpoints x_i, since it doesn't change the integral. Thus, "not in the closure of A_{nm}" is just shorthand for not in A_{nm} and not an endpoint x_i.

or

$$\text{(a)} \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} f(k/n), \quad \text{(b)} \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} g(k/n), \quad \text{(c)} \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} h(k/n)$$

where $f(x) = \frac{1}{x+1}$, $g(x) = \frac{1}{x^2+1}$, and $h(x) = x^3\cos(4x)$, all of which are continuous. Thus, by A.2, these limits are

$$\text{(a)} \int_0^1 \frac{1}{x+1}dx = \log(1+x)\big|_0^1 = \log 2$$

$$\text{(b)} \int_0^1 \frac{1}{x^2+1}dx = \tan^{-1} x\big|_0^1 = \frac{\pi}{4}$$

$$\text{(c)} \int_0^1 x^3\cos(4x)dx = \frac{x^3}{4}\sin(4x)\bigg|_0^1 - \frac{3}{4}\int_0^1 x^2\sin(4x)dx = \frac{\sin 4}{4} + \frac{3x^2}{16}\cos(4x)\bigg|_0^1 - \frac{3}{8}\int_0^1 x\cos(4x)dx$$

$$= \frac{\sin 4}{4} + \frac{3\cos 4}{16} - \frac{3x}{32}\sin(4x)\bigg|_0^1 + \frac{3}{32}\int_0^1 \sin(4x)dx = \frac{5\sin 4}{32} + \frac{21\cos 4}{128} + \frac{3}{128}$$

□

A.4 Some Riemann sums III. *Let n be an integer ≥ 2. If $1 \leq i \leq n$, we write $n = q_{i,n}i + r_{i.n}$, where $q_{i,n}, r_{i,n}$ are integers and $0 \leq r_{i,n} \leq i-1$. Let a_n be the number of i's with $1 \leq i \leq n$ such that $r_{i,n} \geq i/2$. Let $\phi : I \to \mathbb{R}$ be defined by $\phi(0) = 0$ and $\phi(x) = \lfloor 2/x \rfloor - 2\lfloor 1/x \rfloor$.*

(1) *prove that $q_{i,n}$ and $r_{i,n}$ are uniquely determined;*

Solution. Suppose we had $q'_{i,n}$ and $r'_{i,n}$ such that $n = q'_{i,n}i + r'_{i,n}$. Then

$$0 = (q_{i,n} - q'_{i,n})i + r_{i,n} - r'_{i,n} \Rightarrow (q_{i,n} - q'_{i,n}) = r'_{i,n} - r_{i,n}$$

the left hand side is divisible by i, so the right must as well; since $0 \leq r_{i,n}, r'_{i,n} \leq i-1$, this is only possible if $r_{i,n} = r'_{i,n}$. We must then have $q_{i,n} = q'_{i,n}$. □

(2) *prove that ϕ is a ruled function on any interval* $[1/k, 1]$

Solution. It is not true that ϕ is ruled on $[0,1]$, for while $\phi\left(\frac{1}{n+1/2}\right) = 1$ for all n, $\phi\left(\frac{1}{n}\right) = 0$ for all n, and therefore ϕ does not have a right limit at 0, contrary to the third problem of the midterm. But on any interval $[1/k, 1]$, ϕ is a step function and therefore ruled. Indeed, $\phi(x) = 0$ if $n \leq 1/x < n + 1/2$ for some n while $\phi(x) = 1$ if $n + 1/2 \leq 1/x < n+1$, so

$$\phi(x) = \begin{cases} 0 & \frac{1}{n+1/2} < x \leq \frac{1}{n}, \text{ for some } n \in \mathbb{Z}_{\geq 1} \\ 1 & \frac{1}{n+1} < x \leq \frac{1}{n+1/2}, \text{ for some } n \in \mathbb{Z}_{\geq 1} \end{cases} \tag{2}$$

□

(3) *compute $\int_{1/k}^1 \phi$ if $k \geq 2$ and use this to compute $\int_0^1 \phi$;*

Solution. According to equation (2),

$$\int_{1/k}^1 \phi = \sum_{i=1}^{k-1}\left(\frac{1}{i+1/2} - \frac{1}{i+1}\right) = \sum_{i=1}^{k-1}\left(\frac{2}{2i+1} - \frac{1}{i+1}\right) \tag{3}$$

This can be rewritten as

$$2\sum_{i=1}^{k}\frac{1}{k+i} - 1$$

and thus, as an improper integral,

$$\int_0^1 \phi = \lim_{k\to\infty}\int_{1/k}^1 \phi = -1 + 2\lim_{k\infty}\sum_{i=1}^{k}\frac{1}{k+i} = 2\log 2 - 1$$

by problem A.2. □

(4) *prove that $a_n = \sum_{k=1}^n \phi(k/n)$;*

Solution. For a given pair n, i, we have $r_{i,n} \geq 1/2$ iff $q_{i,n} + 1/2 \leq n/i < q_{i,n} + 1$ iff $\phi(i/n) = 1$ from the above. Thus, the number of i, $1 \leq i \leq n$, for which $r_{i,n} \geq i/2$ is $a_n = \sum_{i=1}^{n} \phi(i/n)$. □

(5) *prove that* $\lim_{n\to+\infty} a_n/n$ *exists and compute it;*

Solution. If ϕ were ruled on $[0, 1]$, then by A.2, the limit

$$\int_0^1 \phi = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi(k/n) = \lim_{n\to\infty} a_n/n$$

would exist; by part 3, the limit is, in fact,

$$\lim_{n\to\infty} a_n/n = 2 \log 2 - 1$$

But ϕ is only ruled on $[1/k, 1]$, so equation (4) is not exact, but differs from the above sum by no more than $2/k$, since $\phi \leq 1$ on $[0, 1]$. More precisely, for any $\epsilon > 0$, there exists N such that for all $n > N$,

$$\left| \int_{1/k}^1 \phi - \frac{1}{n} \sum_{i=1}^{n} \phi(i/n) - \frac{2}{k} \right| < \epsilon$$

and taking the limit as $n \to \infty$, and then as $k \to \infty$,

$$2 \log 2 - 1 = \int_0^1 \phi = \lim_{n\to\infty} a_n/n$$

□