Problem Set #5 Part C
Official Solutions

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(10) 1. First let $k = \min\{f(0), f(1)\}$, and let $x \in [0,1]$ be such that $f(x) = 0$. Then by the intermediate value theorem, there exists $x_1 \in [0,x]$ and $x_2 \in [x,1]$ such that $f(x_1) = k/2$ and $f(x_2) = k/2$. However, since f is a bijection we know that $k_1 = k_2$. The only way that this is possible is if $k = 0$, so either $f(0) = 0$ or $f(1) = 0$. A similar argument with $k = \max\{f(0), f(1)\}$ shows that $f(0) = 1$ or $f(1) = 1$.

First assume that $f(0) = 0$. We will show that f is strictly increasing by contradiction. Suppose that there exist $x, y \in [0,1]$ such that $x < y$ but $f(x) \geq f(y)$. Then by the intermediate value theorem there exist $x_1 \in [0,x]$ and $x_2 \in [x,y]$ such that $f(x_1) = (f(x) + f(y))/2$ and $f(x_2) = (f(x) + f(y))/2$. However, this contradicts the fact that f is a bijection. Thus such x, y do not exist, so f must be strictly increasing.

Now suppose that $f(0) = 1$. Let $g(x) = f(1-x)$. Then g is a continuous bijection with $g(0) = 0$, so it is strictly increasing. Thus f is strictly decreasing. So we are done.

(12) 2.

(4) (1) First we will show that f is surjective. Let $x \in [0,1]$, and let $y = f^{n_x - 1}(x)$. Clearly, $y \in [0,1]$. Then, by definition $f(y) = x$, so f is surjective. Now we will show that f is injective. Suppose that $f(x) = f(y)$. Then $f^{n_x}(x) = x$, so $f^{n_x a}(x) = x$ for all $a \in \mathbf{N}$. Similarly, $f^{n_y a} = y$. Thus $f^{n_x n_y}(x) = x$ and $f^{n_x n_y}(y) = y$ so $x = y$. Thus f is injective.

Since f is injective and surjective it is a bijection.

(8) (2) We will show that $f(x) = x$. If f is continuous and $f(0) < 1$ we know (by the previous problem) that $f(0) = 0$ and $f(1) = 1$ and f is strictly increasing. Now suppose that there is some $x \in [0,1]$ such that $f(x) < x$. Consider the sequence $\{x, f(x), f^2(x), \ldots, f^{n_x - 1}(x), x, \ldots\}$. We know that there must be some element y of this sequence such that $f(y) > y$ (otherwise it would be a strictly decreasing, which would be a contradiction). Consider the first such element in the sequence. Thus we have three elements y_1, y_2, y_3 such that $y_1 > y_2$ and $y_2 < y_3$ and $y_2 = f(y_1)$ and $y_3 = f(y_2)$. Then we have that $y_2 < y_1$ but $f(y_1) < f(y_2)$.

This contradicts the fact that f is strictly increasing. Thus there must not exist such an x. Thus for all $x \in [0, 1]$, $f(x) \geq x$. A similar argument shows that for all $x \in [0, 1]$, $f(x) \leq x$. Thus $f(x) = x$.

(18) 3.

(4) (1) Note that if $f(x + \lambda) = f(x)$ then $f(x + n\lambda) = f(x)$ for all $n \in \mathbf{Z}$. Let $\alpha = p/q$. Then $g(x + q\alpha) = g(x + p)$ and $f(x + p \cdot 1) = f(x)$. Thus $(f + g)(x + p) = f(x + p) + g(x + p) = f(x) + g(x) = (f + g)(x)$ so $f + g$ is periodic.

(7) (2) We will show that if $\alpha \in \mathbf{R}\backslash\mathbf{Q}$ then for all $y \in \mathbf{R}$ and all $\epsilon > 0$ there exists $x \in \mathbf{R}$ such that $h(x) = h(0)$ and $|x - y| < \epsilon$.

Since α is irrational we know that there exist $p, q \in \mathbf{Z}$ such that $|q\alpha - p| < \epsilon$. Let $\beta = q\alpha - p$. Then we know that $h(\beta + x) = h(x - \beta) = h(x)$. Then we have $h(\lfloor y/\beta \rfloor \beta) = h(0)$ and $|y - \beta \lfloor y/\beta \rfloor| < \beta < \epsilon$. Then, since h is continuous we will have that $h(y) = h(0)$. Thus h is constant.

(7) (3) We have already show that if $\alpha \in \mathbf{Q}$ then $f + g$ is periodic. Thus all we have left to show is that if $\alpha \notin \mathbf{Q}$ then $f + g$ is not periodic. Suppose that it is. Then there is some $\lambda \in \mathbf{R}$ such that $(f + g)(\lambda + x) = (f + g)(x)$. Thus $f(x + \lambda) - f(x) = g(x) - g(x + \lambda)$. Note that this function is periodic with two periods: 1 and α. Thus it is constant. Let $a = f(x + \lambda) - f(x)$. Then we see that $f(x + n\lambda) = f(x) + na$. Let $M = \sum_{x \in \mathbf{R}} |f|$ (which exists because f is periodic), and let n be such that $na > 2M$. Then $f(x + n\lambda) \leq M \leq f(x) + na = f(x + n\lambda)$, a contradiction. The only way this can be avoided is if $a = 0$. However, if $a = 0$ then f and g must both have period λ, which means that λ is both rational and irrational, another contradiction. So $f + g$ cannot be periodic if $\alpha \in \mathbf{R}\backslash\mathbf{Q}$.

So $f + g$ is periodic if and only if $\alpha \in \mathbf{Q}$.