Solution Set 5: Part B

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B.1. Simple Convergence and uniform convergence I. *Let $I = [a, b]$ and $f_n : I \to \mathbb{R}$ be a sequence of functions.*

(1) *Prove that if f_n converges uniformly to $f : I \to \mathbb{R}$, then it converges simply.*

Solution. This is a one-liner: for any $y \in I$, if we are given $\epsilon > 0$, there exists N such that $|f_n(y) - f(y)| \leq \sup_{x\in I} |f_n(x) - f(x)| < \epsilon$ for all $n > N$, and thus $f_n(y) \longrightarrow f(y)$. □

(2) *Prove that the converse is not true by considering the sequence $f_n : [0, 1] \to \mathbb{R}$,*

$$f_n(x) = x^n \text{ and } f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$$

Solution. First note that f_n converges simply to f; since $f_n(1) = 1$ and $f_n(0) = 0$ for all n, $f_n(1) \longrightarrow f(1)$ and $f_n(0) \longrightarrow 0$, while for all $x \in (0, 1)$, $f_n(x) = x^n \longrightarrow 0 = f(x)$ as $n \longrightarrow \infty$. (For any $\epsilon > 0$, if $x \in (0, 1)$ we may take $N = \lceil \log_x \epsilon \rceil$, in which case $x^n < \epsilon$ for all $n > N$.)

If we take $\epsilon = 1/4$ and $x_n = (1/2)^{1/n}$, then for any N, $|f_N(x_N) - f(x_N)| = 1/2 > \epsilon$, and f_n cannot converge uniformly to f. Indeed, for any n, $\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1$. □

(3) *Prove that even if f is continuous then simple convergence does not necessarily imply uniform convergence by considering*

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq 1/(2n) \\ 1 - nx & \text{if } 1/(2n) \leq x \leq 1/n \\ 0 & \text{if } 1/n \leq x \leq 1 \end{cases}$$

Solution. f_n converges simply to $f = 0$; $f_n(0) = 0$ for all n, so $f_n(0) \longrightarrow 0$, and for all $x \in (0, 1]$, there exists an N such that $1/n < x \Rightarrow f_n(x) = 0$ for all $n > N$, and thus $f_n(x) \longrightarrow 0$.

The convergence is not uniform, however. If we take $\epsilon = 1/4$, then for all N, $|f_N(1/(2N)) - f(1/(2N))| = 1/2$, so that $\sup_{x\in[0,1]} |f_n(x) - f(x)| \geq 1/2$ for all n. (In fact the supremum is always equal to $1/2$.) □

B.2. Simple Convergence and uniform convergence II. *Assume $f_n : I \to \mathbb{R}$ is a sequence of continuous functions converging simply to a continuous function $f : I \to \mathbb{R}$, such that $f_{n+1}(x) \geq f_n(x)$ for every $n \geq 1$, $x \in I$.*

(1) *Show that $y_n = \sup_{x\in I} |f_n(x) - f(x)|$ is non-increasing.*

Solution. For any fixed x, $f_n(x)$ is a non-decreasing convergent sequence, and its limit $f(x)$ must therefore by the supremum of $\{f_n(x)\}_{n\geq 1}$ (which must exist): in particular, $f(x) \geq f_n(x)$ for all n. Thus, $f(x) \geq f_{n+1}(x) \geq f_n(x)$ for all $n \Rightarrow |f(x) - f_{n+1}(x)| \leq |f(x) - f_n(x)|$ for all n and $x \in I \Rightarrow y_{n+1} \leq y_n$. □

(2) *Prove that f_n converges uniformly to f.*

Solution. (adapted from Parker Meares' solution.) y_n is a non-increasing sequence bounded from below, since $y_n \geq 0$ for all n, so it converges to the infimum of $\{y_n\}_{n\geq 1}$. Suppose the limit is $2\epsilon > 0$. Let $h_n(x) = |f_n(x) - f(x)|$, and let $A_n \subset I$ be the set of all x such that $h_n(x) \geq \epsilon$; A_n is nonempty since $\sup_{x\in I} h_n(x) > \epsilon$. h_n is continuous, so $h_n^{-1}(B(0,\epsilon))$ is open; its complement, A_n, is a closed subset of the compact interval I and is itself compact. $f_{n+1}(x) \geq f_n(x) \Rightarrow h_{n+1}(x) \geq h_n(x)$ for any n, so $A_{n+1} \subset A_n$. For any $x \in I$, $h_n(x) \longrightarrow 0$, so for some N we must have $x \notin A_n$ for $n > N$ and therefore $x \notin \cap_{n\geq 1} A_n$. It then follows that $\cap_{n\geq 1} A_n = \varnothing$, contradicting problem C.1 on problem set 3 (the intesection of a countable nested sequence of compact sets is nonempty). □

B.3. Simple Convergence and uniform convergence III. *Assume $f_n : I \to \mathbb{R}$ is a sequence of functions converging simply to $f : I \to \mathbb{R}$, such that for any $x \in I$ and any sequence $x_n \longrightarrow x$, $f_n(x_n) \longrightarrow f(x)$.*

(1) *Prove that for every $\epsilon > 0$ and $x \in I$, there exists $\delta > 0$ and N such that $y \in B(x,\delta) \Rightarrow f_n(y) \in B(f_n(x))$ for all $n > N$.*

Solution. Suppose not; assume that for some ϵ, given any δ and N there is some $y \in B(x,\delta)$ for which $f_n(y)$ is not within ϵ of $f_n(x)$ for some $n > N$. This implies that for all i there is some $\varphi(i) > i$ and $y_{\varphi(i)} \in B(x, 1/i)$ such that $f_{\varphi(i)}(y_{\varphi(i)}) \notin B(f_{\varphi(i)}(x), \epsilon)$. Note that we can always choose $\varphi(i)$ greater than all previously chosen $\varphi(j)$, $j < i$. $f_n(x) \longrightarrow f(x)$ by simple convergence, so for some N, $f_n(x) \in B(f(x), \epsilon/2)$ for all $n > N$. By the triangle inequality, for any $z \in B(f(x), \epsilon/2)$,

$$d(z, f_n(x)) \leq d(z, f(x)) + d(f(x), f_n(x)) < \epsilon \Rightarrow B(f(x), \epsilon/2) \subset B(f_n(x), \epsilon) \quad \forall n > N$$

$y_{\varphi(i)}$ converges to x by construction, but $f_{\varphi(i)}(y_{\varphi(i)})$ is not in $B(f(x), \epsilon/2)$ for any $\varphi(i) > N$, and therefore cannot converge to $f(x)$, a contradiction. □

(2) *Prove that f is continuous.*

Solution. For all $\epsilon > 0$, there exists $\delta > 0$ and M such that for all $x, y \in I$, $|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon/3$ for all $n > M$ by the above. By simple convergence, for each such pair x, y, there exists N, P such that $|f_n(x) - f(x)| < \epsilon/3$ for $n > N$, and $|f_n(y) - f(y)| < \epsilon/3$ for $n > N$, so that

$$|f(x) - f(y)| < |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \epsilon$$

(for all $n > \max\{M, N, P\}$, though we don't need this any more). □

(3) *Prove that f_n converges uniformly to f.*

Solution. By the simple convergence of f_n, the continuity of f, and part (1), for any $x \in I$ and $\epsilon > 0$, there exists $\delta_x > 0$ and N_x such that for any $y \in B(x,\delta)$ and $n > N_x$, $|f_n(x) - f(x)| < \epsilon/6$, $|f(x) - f(y)| < \epsilon/6$ and $|f_n(x) - f_n(y)| < \epsilon/6$, where N_x is obtained by taking the maximum of the N's guaranteed for each inequality, as in (2). Thus,

$$|f(y) - f_n(y)| < |f_n(x) - f(x)| + |f(x) - f(y)| + |f_n(x) - f_n(y)| < \epsilon/2 \qquad \forall y \in B(x, \delta_x), n > N_x$$

I is compact and therefore covered by finitely many balls of radius ϵ; take their centers to be x_i. Then if we let $N = \max N_{x_i}$ and $\delta = \min\{\delta_{x_i}\}$,

$$\begin{aligned} |f(y) - f_n(y)| &< \epsilon/2 \qquad \forall y \in B(x,\delta), n > N \\ \Rightarrow \sup_{y\in I} |f(y) - f_n(y)| &< \epsilon \qquad \forall y \in B(x,\delta), n > N \end{aligned}$$

and f_n uniformly converges to f. □