HOMEWORK 5, PART A

MATH 25

1. Problems graded by Jennifer

1.1. **The AM-GM inequality.** *Let $a_1, \cdots, a_n$ be positive real numbers. The arithmetic mean of the a_k's is $(a_1 + \cdots + a_n)/n$ and their geometric mean is $\sqrt[n]{a_1 \cdots a_n}$. The AM-GM inequality says that*

$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

The goal of this problem is to prove the above inequality. If $a, b \geq 0$, compute $(\sqrt{a} + \sqrt{b})^2 - (\sqrt{a} - \sqrt{b})^2$ and deduce that $\sqrt{ab} \leq (a + b)/2$

Proof: We have $(\sqrt{a} + \sqrt{b})^2 - (\sqrt{a} - \sqrt{b})^2 = 4\sqrt{ab}$ and since $(\sqrt{a} + \sqrt{b})^2 - (\sqrt{a} - \sqrt{b})^2 \leq (\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$, we have $\sqrt{ab} \leq (a + b)/2$.

Use this to prove that if $k \geq 1$ and $a_1, \cdots, a_{2^k}$ are positive real numbers, then

$$\sqrt[2^k]{a_1 \cdots a_{2^k}} \leq \frac{a_1 + \cdots + a_{2^k}}{2^k}.$$

Proof: For $k = 1$, this is the above result. Now if we set $a = \sqrt[2^{k-1}]{a_1 \cdots a_{2^{k-1}}}$ and $b = \sqrt[2^{k-1}]{a_{1+2^{k-1}} \cdots a_{2^k}}$, then $\sqrt{ab} \leq (a + b)/2$ so that

$$\sqrt[2^k]{a_1 \cdots a_{2^k}} \leq (a + b)/2 \leq \frac{\sqrt[2^{k-1}]{a_1 \cdots a_{2^{k-1}}} + \sqrt[2^{k-1}]{a_{1+2^{k-1}} \cdots a_{2^k}}}{2} \leq \frac{a_1 + \cdots + a_{2^k}}{2^k}$$

prove the AM-GM inequality. Hint: given $a_1, \cdots, a_n$ choose k such that $2^k \geq n$ and set $a_{n+1}, \cdots, a_{2^k} = \sqrt[n]{a_1 \cdots a_n}$

Proof: Set $a = a_{n+1}, \cdots, a_{2^k} = \sqrt[n]{a_1 \cdots a_n}$. We then have

$$\sqrt[2^k]{a_1 \cdots a_{2^k}} \leq \frac{a_1 + \cdots + a_{2^k}}{2^k}$$

which implies that

$$\sqrt[2^k]{a_1 \cdots a_n a^{2^k - n}} \leq \frac{a_1 + \cdots + a_n + (2^k - n)a}{2^k}$$

and $a_1 \cdots a_n = a^n$ so that

$$a \leq \frac{a_1 + \cdots + a_n + (2^k - n)a}{2^k}$$

and this directly implies

$$\sqrt[n]{a_1 \cdots a_n} \le \frac{a_1 + \cdots + a_n}{n}.$$

1.2. *Let $\{f_n\}_{n\ge 0} : \mathbf{R}_{\ge 0} \to \mathbf{R}$ be the sequence of functions defined by $f_0(x) = 0$ and*

$$f_{n+1}(x) = f_n(x) + \frac{1}{2}(x - f_n^2(x)).$$

Prove that for every $x \in [0,1]$,

$$0 \le \sqrt{x} - f_n(x) \le \sqrt{x}\left(1 - \frac{\sqrt{x}}{2}\right)^n.$$

Proof: for $n = 0$, the above inequality is just $0 \le \sqrt{x} \le \sqrt{x}$ which is certainly true. Now assuming that it is true for n, we will prove it for $n+1$. Note that $\sqrt{x} - f_n(x) \le \sqrt{x}\left(1 - \frac{\sqrt{x}}{2}\right)^n$ implies that $f_n(x) \ge 0$. We have

$$\begin{aligned}
\sqrt{x} - f_{n+1}(x) &= \sqrt{x} - f_n(x) - \frac{1}{2}(x - f_n^2(x)) \\
&= (\sqrt{x} - f_n(x))\left(1 - \frac{1}{2}(\sqrt{x} + f_n(x))\right) \\
&\le \sqrt{x}\left(1 - \frac{\sqrt{x}}{2}\right)^n \left(1 - \frac{1}{2}(\sqrt{x} + f_n(x))\right) \\
&\le \sqrt{x}\left(1 - \frac{\sqrt{x}}{2}\right)^n \left(1 - \frac{\sqrt{x}}{2}\right) \\
&= \sqrt{x}\left(1 - \frac{\sqrt{x}}{2}\right)^{n+1}
\end{aligned}$$

while

$$\begin{aligned}
\sqrt{x} - f_{n+1}(x) &= (\sqrt{x} - f_n(x))\left(1 - \frac{1}{2}(\sqrt{x} + f_n(x))\right) \\
&\ge (\sqrt{x} - f_n(x))\left(1 - \frac{1}{2}(2\sqrt{x})\right) \\
&\ge 0.
\end{aligned}$$

Prove that f_n converges uniformly to $\sqrt{x}$ on $[0,1]$.

Proof: the function $y \mapsto y(1 - y/2)$ from $[0,1]$ to $\mathbf{R}$ has its maximum is at $y = 1$ where it is equal to $1/2$. This implies that

$$0 \le \sqrt{x} - f_n(x) \le \left(\frac{1}{2}\right)^n$$

and therefore that f_n converges uniformly to $\sqrt{x}$ on $[0,1]$.