Solution Set 4: Part B

Benjamin Bakker
Mathematics 25a
Prof. Berger

October 17, 2003

B.1. Writing a number in base b. *Let $b \geq 2$ be an integer, and choose $n \in \mathbb{N}$.*

(1) *Show that one can write $n = \sum_{k=0}^{\ell} x_k b^k$ with $x_k \in \{0, 1, \ldots, b-1\}$ and some $\ell \geq 0$ in exactly one way.*

Solution. We prove the claim by induction on the highest power $\ell \in \mathbb{N}$ such that $b^\ell \leq n$. The only case in which there is no such ℓ is $n = 0$, where $n = 0$ is the unique expansion. For $\ell = 0$, we have $1 \leq n \leq b-1$, and so the unique expansion is trivially $n = nb^0$. If $\ell \geq 1$, assume the result for all $\ell' < \ell$. If $b^\ell \leq n < b^{\ell+1}$, so that $1 \leq nb^{-\ell} < b$, there is a unique integer $x_\ell \in \{1, \ldots, b-1\}$ such that $nb^{-\ell} \in [x_\ell, x_\ell + 1)$ (that is, $x_k = \lfloor nb^{-\ell} \rfloor$). Thus, $n - x_\ell b^\ell < b^\ell$, and so by hypothesis there is a unique expansion $n - x_\ell b^\ell = \sum_{k=0}^{\ell'} x_k b^k$ for some $\ell' < \ell$. Defining $x_k = 0$ for $k > \ell'$ and $k \neq \ell$, the unique expansion of n is $n = \sum_{k=0}^{\ell} x_k b^k$. □

(2) *Let $x \in [0, 1)$. Show that one can write $x = \sum_{k=1}^{\infty} x_k b^{-k}$ with $x_k \in \{0, 1, \ldots, b-1\}$ in exactly one way, if we impose the additional condition that for every $N \geq 0$, there is an $n \geq N$ such that $x_n \neq b - 1$.*

Solution. Let $x_0 = 0$, and following the pattern in the above, inductively define $y_{k-1} = \sum_{i=1}^{k-1} x_i b^{-i}$ ($y_0 = 0$) to be the $(k-1)$th partial sum and $x_k = \lfloor (x - y_{k-1}) b^k \rfloor$, $k \geq 1$. Thus, for any k, $x_k b^{-k} \leq x - \sum_{i=0}^{k-1} x_i b^{-i} < (x_k + 1) b^{-k}$, and

$$|x - y_k| < b^{-k}$$

The series y_k obviously converges since $b \geq 2$ and $b^{-k} \longrightarrow 0$. If $x_k = b - 1$ for all $k > m$ for some m, then the limit of the above series starting at $m + 1$ is b^{-m} (see below), and therefore $x - y_m = b^{-m}$, contrary to the above. Thus, we cannot have infinite strings of $b - 1$.

Assume that there are two such series, $x = \sum_{k \geq 1} x_k b^{-k} = \sum_{k \geq 1} x'_k b^{-k}$. Suppose that they first differ at the ℓth digit, and that the x'_k expansion has the higher digit. Then, since the difference of two convergent series converges to the difference of the limits,

$$0 = \sum_{k \geq 1} (x'_k - x_k) b^{-k} = (x'_\ell - x_\ell) b^{-\ell} + \sum_{k > \ell} (x'_k - x_k) b^{-k}$$

$$\Rightarrow (x'_\ell - x_\ell) b^{-\ell} = \sum_{k > \ell} (x_k - x'_k) b^{-k} < \sum_{k > \ell} (b-1) b^{-k}$$

where we have <u>strict</u> inequality because of the added condition that there are no infinite strings of $b - 1$'s. But the series on the right is

$$\sum_{k > \ell} (b-1) b^{-k} = (b-1) b^{-\ell-1} \sum_{k \geq 0} b^{-k} = b^{-\ell-1} \frac{b-1}{1 - b^{-1}} = b^{-\ell} \Rightarrow b^{-\ell} < b^{-\ell}$$

and we have a contradiction, so the expansion is unique. □

B.2. The digits of 2^n. *Let $\log_2$ stand for the logarithm base 2: if $x > 0$, $\log_2 x$ is the unique real number such that $2^{\log_2 x} = x$.*

(1) *Prove that $\log_2 10$ is irrational.*

Solution. Suppose it was rational, $\log_2 10 = p/q$ for some $p, q \in \mathbb{Z}_{\geq 1}$. Then $2^p = 10^q = 2^q 5^p$; if $p > q \geq 1$ then 2 either divides 5 or 1, while if $1 \leq p \leq q$, either 5 divides 1 or $p = q = 1 \Rightarrow 2 = 10$. All possibilities lead to a contradiction, so $\log_2 10$ is irrational. $\square$

(2) *Show that there is a number $n \geq 1$ such that when one writes 2^n in base 10, the first few digits are 12121976....*

Solution. In other words, we must prove the existence of an integer n such that

$$12121976 \times 10^m \leq 2^n < 12121977 \times 10^m$$
$$\iff m \log_2 10 + \log_2(12121976) \leq n < m \log_2 10 + \log_2(12121977)$$

for some positive integer m. We essentially proved in lecture (as a Corollary to Dirichlet's Theorem) that the fractional part of the positive multiples of an irrational number are dense in $(0, 1)$. By (1), $-\log_2 10$ is irrational, so for x the midpoint of $\log_2(12121976)$ and $\log_2(12121977)$, and ϵ half of the difference between them, there exists an integer $m \in \mathbb{Z}_{\geq 1}$ such that $|\{-m \log_2\} - x| < \epsilon$. But $\{-m \log_2 10\} = 1 - \{\log_2 10\}$, and therefore for some positive integer $n - 1$ (I'm calling this $n - 1$ just so the answer is cleaner),

$$\log_2(12121976) < 1 - (m \log_2 10 - (n-1)) < \log_2(12121977)$$

2^n then begins with the digits 12121976 (and has $m + 8$ digits altogether). $\square$

Note: James Dowdell found that one such number is $n = 4205827$. The corresponding power, 2^n, has 1266081 digits that fill roughly 353 pages. This problem can be generalized; you can find a power of 2 whose base ten expansion begins with any string of digits you choose.

B.3. Lévy's chords lemma. *Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$ and choose $q \in \mathbb{Z}_{\geq 1}$. Prove that there exists $x \in [0, 1 - 1/q]$ such that $f(x + 1/q) = f(x)$.*

Solution (adapted from Eric Suh's proof). Let $g(x) = f\left(x + \frac{1}{q}\right) - f(x)$. Consider the values of g on the points n/q, with $n \in \{0, 1, \ldots, q-1\}$. The sum

$$\sum_{n=0}^{q-1} g(n/q) = f(1/q) - f(0/q) + f(2/q) - f(1/q) + \cdots + f(q/q) - f(1 - 1/q) = f(1) - f(0) = 0$$

vanishes by assumption. Thus, either $g(n/q) = 0$ for all such n, in which case $x = 0$ satisfies the desired condition, or for some $m, n < q$, $g(m/q) > 0$ and $g(n/q) < 0$ (assume $m > n$). By the intermediate value theorem, there is some $x \in \left(\frac{n}{q}, \frac{m}{q}\right) \subset \left[0, 1 - \frac{1}{q}\right]$ such that $g(x) = 0 \Rightarrow f\left(x + \frac{1}{q}\right) = f(x)$.

$\square$