# SOLUTIONS PART A

## A. Math 25a: Homework 4A

A.1. **A regularization operator.** Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers. Set

$$y_n = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

If $y_n \to y$ for some $y \in \mathbb{R}$, we say that $\{x_n\}_{n\geq 1}$ Cesàro-converges to $y$.

(1) Show that if $x_n$ converges to $x$, then it also Cesàro-converges to $x$.
(2) Find the Cesàro-limit of the sequence $\{x_n = (-1)^n\}$.

*1.* If $\{x_n\}_{n\geq 1}$ converges, then it is also bounded and we fix $M$ such that $|x_n - x| \leq M$ for all $n \geq 1$. Choose $\varepsilon > 0$ and let $N$ be such that $|x_n - x| < \varepsilon$ if $n \geq N$. For $m \geq 1$, we have

$$\begin{aligned}
|y_{N+m} - x| &= \left|\frac{1}{N+m}(x_1 + \cdots + x_{N+m}) - x\right| \\
&= \left|\frac{1}{N+m}((x_1 - x) + \cdots + (x_{N+m} - x))\right| \\
&= \left|\frac{1}{N+m}((x_1 - x) + \cdots + (x_N - x)) + \frac{1}{N+m}((x_{N+1} - x) + \cdots + (x_{N+m} - x))\right| \\
&\leq \frac{1}{N+m}(MN) + \frac{1}{N+m}(m\varepsilon)
\end{aligned}$$

This shows that for any $m$ such that $MN/(N+m) < \varepsilon$, we have $|y_{N+m} - x| < \varepsilon$ which shows the convergence of $y_n$ to $x$. $\square$

*2.* Observe that for $n \geq 1$, $x_{2n+1} = -1$ and $x_{2n} = 1$, which means that

$$y_m = \begin{cases} -\frac{1}{m} & \text{for } m = 2n+1 \\ 0 & \text{for } m = 2n. \end{cases}$$

Hence, the Cesàro-limit of the sequence $\{x_n = (-1)^n\}$ is 0. $\square$

A.2. **Extending continuous functions.** Let $f : [0,1) \to \mathbb{R}$ be a continuous function. Show that one can extend $f$ to a continuous function on $[0,1]$ if and only if $f$ is uniformly continuous on $[0,1)$.

*Proof.* If $f$ extends to a continuous function $f : [0,1] \to \mathbb{R}$, then $f$ is uniformly continuous on $[0,1]$ by Heine's theorem, since $[0,1]$ is compact. Its restriction to $[0,1)$ will also be uniformly continuous then.

Suppose now that $f$ is uniformly continuous on $[0,1)$. We need to define $f(1)$ so that the extended function will be continuous. Note that we proved this in class in more generality! We'll do it here anyway. We claim:

(1) if $(s_n)$ is a sequence in $[0,1)$ converging to 1, then $\{f(s_n)\}$ converges;
(2) if $\{s_n\}$ and $\{t_n\}$ are sequences in $[0,1)$ converging to 1, then $\lim f(s_n) = \lim f(t_n)$.

We will prove this claim below, but observe first that if it is true, then one can define $f(1)$ as the limit of any sequence $\{f(s_n)\}$ where $(s_n)$ is a sequence in $[0,1)$ converging to 1. The resulting $f : [0,1] \to \mathbb{R}$ will be continuous by the sequential definition for continuity.

To prove (1), note that $\{s_n\}$ is a Cauchy sequence, and since $f$ is uniformly continuous, $\{f(s_n)\}$ is also a Cauchy sequence (see the more general proof given in class). Hence $\{f(s_n)\}$ converges.

To prove (2), we define a third sequence $\{u_n\}$ such that $\{s_n\}$ and $\{t_n\}$ are both subsequences of $\{u_n\}$. This is easily accomplished by interlacing $\{s_n\}$ and $\{t_n\}$:

$$\{u_n\}_{n=1}^{\infty} = \{s_1, t_1, s_2, t_2, s_3, t_3, \ldots\}.$$

We see that $\lim u_n = 1$, which implies that $\lim f(u_n)$ exists. The two subsequences $\{f(s_n)\}$ and $\{f(t_n)\}$ of $\{f(u_n)\}$ both must converge to $\lim f(u_n)$, so $\lim f(s_n) = \lim f(t_n)$. This proves the two claims. □

A.3. **Discontinuities of non-decreasing functions.** Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function.

(1) Show that $f$ is continuous at $x \in \mathbb{R}$ if and only if $\sup_{y<x} f(y) = f(x)$ and $\inf_{y>x} f(y) = f(x)$ (draw a picture!).
(2) Prove that the set of $x \in \mathbb{R}$ at which $f$ is not continuous is countable.

*1.* ($\Rightarrow$) This implication does not use the fact that $f$ is non-decreasing. Suppose that $f$ is continuous at $x \in \mathbb{R}$. Then for any sequence $x_n \to x$, we have $f(x_n) \to f(x)$, so if $x_n$ is a sequence of real numbers such that $f(x_n) \to \inf_{y>x} f(y)$, then $f(x_n) \to f(x)$ and therefore $f(x) = \inf_{y>x} f(y)$. The exact same argument will show that $\sup_{y<x} f(y) = f(x)$.

($\Leftarrow$) Let $f$ be a non-decreasing function and $x \in \mathbb{R}$ such that $\sup_{y<x} f(y) = f(x)$ and $\inf_{y>x} f(y) = f(x)$. We will prove that $f$ is continuous at $x$. Choose $\varepsilon > 0$. Because $\sup_{y<x} f(y) = f(x)$, there exists $z < x$ with $f(x) > f(z) > f(x) - \varepsilon$. Similarly, because

$\inf_{y>x} f(y) = f(x)$, there exists $w > x$ with $f(x) < f(w) < f(x) + \varepsilon$. Since $f$ is non-decreasing, we see that for all $y$ in the interval $(w, z)$ we must have $|f(y) - f(x)| < \varepsilon$. This shows that $f$ is continuous. $\square$

*2.* If $f$ is discontinuous at $x$, then choose a rational number $q_x$ in the interval

$$(\sup_{z<x} f(z), \inf_{y>x} f(y)).$$

This defines an injective map from the set of discontinuities of $f$ to the rational numbers, hence the set of discontinuities of $f$ is countable. $\square$