Problem Set #3 Part C
Official Solutions

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(8) 1. Consider the sequence $\{x_n\}$ where $x_n \in F_n$. Now, since E is compact, we know that this sequence has an accumulation point x. Let $\{y_n\}$ be the subsequence that converges to x. Note that for each i, $\{y_n\}_{n=i}^{\infty} \subset F_i$. In addition, since each of the F_i is closed, we know that each contains the limit of $\{y_n\}$, which is x. Thus $x \in F_i$ for all i, so $\{x\} \subset \cap_{n=1}^{\infty} F_n$ so it is nonempty.

Now suppose that E is $(0, 1]$ and let $F_n = (0, 1/n]$. Then for all $x \in (0, 1]$, if we let N be the smallest integer greater than $1/x$ we see that $x \notin F_N$. Thus $\cap_{n=1}^{\infty} F_n$ is empty.

(7) 2. Since E is compact, we know that for every $r > 0$ it can be covered with finitely many open balls of radius r. Let U_n be the set of open balls of radius $1/n$ that cover E. Let A_n be the set of centers of these balls. We know that A_n is finite, so it is countable. In addition, we know that every point in E is within $1/n$ of one of the points of A_n. Let $A = \cup_{n=1}^{\infty} A_n$. This is a countable union of countable sets, so it is countable. Fix $\epsilon > 0$, and let $x \in E$. Let $N > 1/\epsilon$. Then we know that there is a point $y \in A_N$ such that $d(x, y) < \epsilon$, so there is a point $y \in A$ such that $d(x, y) < \epsilon$. Thus A is dense in E. So every compact space has a countable dense subset.

(25) 3.

(5) (1) Consider the complement of K. At the nth step, a union of open sets was taken out of K_n. Thus the complement of K, which is going to be the union of all of the intervals which was taken out, is going to be a union of open sets, which means that it will be open. Thus K is closed. We know that a closed subset of a compact space is compact. Therefore K is compact.

(5) (2) Note that at every step we take out a finite number of intervals. Since we do a countable number of steps, and at each point we take out a countable number of intervals, in the end we will have taken out a countable number of intervals.

Note that at every step every interval present in K_n is reduced by 1/3 of its length. Thus at every step we take out 1/3 of the length that we have not yet taken out. So at the first step we will have taken out a total of $1/3$. At the second step we take out a length of $2/3 \cdot 1/3$. At the nth step, we take out a length of $(2/3)^{n-1} \cdot 1/3$. Thus total, we take out

$$\sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \cdot \frac{1}{1-2/3} = \frac{1}{3}3 = 1.$$

(5) (3) Suppose that the connected component of a point x is some interval in $\mathbf{R}$. Then it will be of some length ϵ. Note, however, that that will mean that there is some interval of length ϵ that has not been taken out, which means that the total length taken out is at most $1-\epsilon \leq 1$. Since we know that $1-\epsilon = 0$ (by part (2)) we see that $\epsilon = 0$, which means that the connected component of the point is just the point itself.

(5) (4) We will prove this by induction. Write the real numbers in $[0,1]$ in base 3. Then $K_0 = [0,1]$. Note that when we take out the middle third of the interval we remove all real numbers such that $a_1 = 1$, except for $1/3$. However, $1/3$ can be written in base 3 with $a_i = 2$ for $i > 1$ and $a_1 = 0$. Thus after one step we have eliminated all points with $a_1 = 1$. Now, suppose that after n steps we have removed all points with $a_i = 1$ for $i \leq n$. Consider the $n+1$-st step. Fix some interval. This will be the set of real numbers α where the first n a_i are a fixed sequence of 0s and 2s, and the rest are allowed to vary. When we remove the middle third of this we will remove all α in this interval such that $a_{n+1} = 1$ other than the one where $a_i = 0$ for all $i > n+1$. However, this α can be written with $a_{n+1} = 0$ and $a_i = 2$ for all $i > n+1$. Thus after $n+1$ steps we will have removed all α with $a_{n+1} = 1$. In addition, we will not have removed any other numbers. Thus K will be the set of all α that can be written in the form $\sum_{i=1}^{\infty} a_i 3^{-i}$ for $a_i \in \{0,2\}$.

(5) (5) We will construct a bijection between the infinite sequences consisting of 0s and 1s, and the points of the Cantor set. Consider any point $x \in K$. Then we can write $x = \sum_{i=1}^{\infty} a_i 3^{-i}$. Map this to the sequence $\{a_i/2\}$, which will consist of 0s and 1s. Note that this is a bijection, since every sequence will yeild such a number, and every number yeilds a sequence. Thus we know that $\mathrm{Card}(K) = \mathrm{Card}(\{0,1\}^{\mathbf{Z}})$, which we know is an uncountable set. Thus we see that K is uncountable.