Solution Set 3: Part B

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B.1. (1) *Show that if $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuous and $f(x) \longrightarrow \ell$ when $x \longrightarrow +\infty$, then f is uniformly continuous.*

Solution. $\forall \epsilon > 0$, because $f(x) \longrightarrow \ell$ as $x \longrightarrow +\infty$, there exists $A \geq 0$ such that $f(x) \in B(\ell, \epsilon/2)$ whenever $x \geq A$. Consider the restriction of f to $[0, A+1] \subset \mathbb{R}_{\geq 0}$. $[0, A+1]$ is compact, so the restriction is uniformly continuous, and therefore there exists $\delta' > 0$ such that for all $x, x' \in \mathbb{R}_{\geq 0}$

$$|x - x'| < \delta' \Rightarrow |f(x) - f(x')| < \epsilon$$

The smaller of δ' and 1 will do the same job, so we may as well let $\delta = \min(\delta', 1)$. For any $x, x' \in [A, \infty)$, we already know that $f(x), f(x') \in B(\ell, \epsilon/2) \Rightarrow |f(x) - f(x')| < \epsilon$. If $|x - x'| < \delta$, then either $x, x' \in [0, A+1]$ or $[A, \infty)$, so it follows that for any $x, x' \in \mathbb{R}_{\geq 0}$,

$$|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \epsilon$$

and f is uniformly continuous. $\square$

(2) *Show that if $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is uniformly continuous then there exist $A, B \in \mathbb{R}$ such that $|f(x)| \leq Ax + B$ for all $x \in \mathbb{R}_{\geq 0}$.*

Solution. Let $\epsilon = 1$, and $B \in \mathbb{R}$ such that $B > |f(0)| + 1$. There then exists $\delta > 0$ such that $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < 1$ for all $x - x'$. Let $A = 2/\delta$. Then for any $x, x' \in [0, \delta/2]$, $|f(x) - f(0)| < 1$ implies that $|f(x)| < 1 + |f(0)| < B$, so that $|f(x)| < B \leq Ax + B$ on $[0, \delta/2]$. Now, assume $|f(n\delta/2)| < A((n-1)\delta/2) + B = (n-1) + B$. Then once again $|f(x) - f(n\delta/2)| < 1$ on $[n\delta/2, (n+1)\delta/2]$, so $|f(x)| < 1 + |f(n\delta/2)| = n + B$ for all $x \in [n\delta/2, (n+1)\delta/2]$, and therefore by induction $|f(x)| < Ax + B$ on $[0, (n+1)\delta/2]$ for any n. Similarly, we may extend the above argument to negative x, and it follows that $|f(x)| < Ax + B$ for all x. $\square$

(3) *Which of the following functions $\mathbb{R}_{\geq 0} \to \mathbb{R}$ is uniformly continuous?*

$$\text{(a)}\ \ x \to e^x \qquad \text{(b)}\ \ x \to \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad \text{(c)}\ \ x \to \sqrt{x}$$

Solution. (a). If e^x was uniformly continuous, then given $\epsilon = 1$ there exists $\delta > 0$ such that for all $x, x' \in \mathbb{R}_{\geq} 0$, $|x - x'| < \delta \Rightarrow |e^x - e^{x'}| < 1$. If this were the case, then for all $x \in \mathbb{R}_{\geq 0}$, $e^{x+\delta/2} - e^x = e^x(e^{\delta/2} - 1) < 1$ and therefore

$$e^x < \frac{1}{e^{\delta/2} - 1}$$

e^x is not bounded, and thus the above cannot be true.

(b). f is continuous at 0 since for any $\epsilon > 0$, if $x - x' < \epsilon/2$, then $|f(x)| \leq \epsilon/2$ and similarly for x', so $|f(x) - f(x')| < \epsilon$; f is evidently continuous everywhere else as well. The limit of f as $x \longrightarrow +\infty$ is 1 (by L'Hopital's rule; alternatively, $\sin x$ is essentially defined by its power series, and $x \sin \frac{1}{x} = 1 - 1/6x^2 + \cdots \longrightarrow 1$), so by (1) f is uniformly continuous.

(c). For all $\epsilon > 0$, let $\delta = \epsilon^2$. For any $x, x' \in \mathbb{R}_{\geq 0}$, by the triangle inequality

$$|\sqrt{x} - \sqrt{x'}| \leq \sqrt{x} + \sqrt{x'}$$

and so

$$(\sqrt{x} - \sqrt{x'})^2 \leq |\sqrt{x} - \sqrt{x'}|(\sqrt{x} + \sqrt{x'}) = |x - x'| < \epsilon^2 \Rightarrow |\sqrt{x} - \sqrt{x'}| < \epsilon$$

$\square$

B.2. *Show that if $F \subset [0,1]$ is closed then there exists a continuous function $f : [0,1] \to \mathbb{R}$ such that $f^{-1}(0) = F$*

Solution. Let $f(x) = d(x, F)$ as defined in A.1. By A.1, f is continuous, and $f(x) = 0 \Leftrightarrow x \in \overline{F} = F$ since F is closed. $\square$

B.3. *Show that if $f : \mathbb{R} \to \mathbb{R}$ is bijective and continuous, then f is a homeomorphism.*

Solution. We need to show that the inverse of f is continuous; since f is bijective, we must have that the image $f(U)$ of any open $U \subset \mathbb{R}$ is open. Let $B = B(x, \epsilon) \subset$ be any ball; B is contained in the compact set $I = [x - \epsilon, x + \epsilon]$. In lecture we showed that a continuous bijection from a compact space is a homeomorphism, so the restriction of f to I is a homeomorphism, and $f(B)$ is open in $f(I)$. $f(I)$ is a connected compact subset of $\mathbb{R}$ (the image of a connected (compact) space under a continuous map is connected (compact)) and is therefore an interval $[a, b]$; neither a nor b is in $f(B)$, for if a were, then $f(B) \setminus \{a\}$ would be connected but $B \setminus \{f^{-1}(a)\}$ would not, and similarly for b. Thus, $f(B)$ is open.
. For any open set $U \subset \mathbb{R}$ and any point $x \in U$, there must be an open ball $B_x \subset U$ centered at x by definition. We then have $\cup_{x \in U} B_x = U$, and since f is a bijection, $f(U) = \cup_{x \in U} f(B_x)$. By the above, each $f(B_x)$ is open and therefore $f(U)$ is open. f is then a homeomorphism. $\square$