A.1 **Distance to a subset.** Let (E, d) be a metric space, and let A be a subset of E. For $x \in E$, let $d(x, A) = \inf_{a \in A} d(x, a)$.

(1) Show that $x \mapsto d(x, A)$ is continuous;

Proof. We will actually prove that $|d(x, A) - d(y, A)| \leq d(x, y)$ for all $x, y \in E$. By symmetry, it suffices to prove that $d(x, A) - d(y, A) \leq d(x, y)$. Choose $\epsilon > 0$ and let $a \in A$ be such that $d(y, A) \leq d(y, a) \leq d(y, A) + \epsilon$. We have $d(x, A) \leq d(x, a)$ and by the triangle inequality, $d(x, a) \leq d(x, y) + d(y, a)$ so that $d(x, A) \leq d(x, y) + d(y, A) + \epsilon$ which implies that $d(x, A) - d(y, A) \leq d(x, y) + \epsilon$. Since this is true for all $\epsilon > 0$, we must have $d(x, A) - d(y, A) \leq d(x, y)$ which is what we wanted to prove. □

(2) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

Proof. We have $d(x, A) = 0$ if and only if there exists a sequence $\{a_n\}_{n \geq 1}$ of A such that $d(x, a_n) \to 0$ or in other words, if and only if and only if there exists a sequence $\{a_n\}_{n \geq 1}$ of A with $a_n \to x$ that is, if and only if $x \in \overline{A}$. □

A.2 **Extrema on compact spaces.** Show that if E is a compact metric space and $f : E \to \mathbb{R}$ is a continuous map, then f admits a maximum on E. That is, there exists $x \in E$ such that $f(x) \geq f(y)$ for any $y \in E$.

Proof. Since E is compact, $f(E) \subset \mathbb{R}$ is the continuous image of a compact set and it is also compact. The compact subsets of $\mathbb{R}$ are the closed and bounded subsets. Since $f(E)$ is bounded, it admits a sup $\mu = f(x) \in \mathbb{R}$ and there exists a sequence $\mu_n \in f(E)$ that converges to it. Since $f(E)$ is closed, the limit of this sequence is also in $f(E)$. Therefore, we have constructed $\mu \in f(E)$ such that $f(y) \leq f(x) = \mu$ for all $y \in E$, meaning that μ is the maximum of f on E. □

Use this to show that if F_1, F_2 are two disjoint closed subsets of E then there exists $\mu > 0$ such that $d(a_1, a_2) \geq \mu$ whenever $a_1 \in F_1$ and $a_2 \in F_2$.

Proof. Let $g : F_1 \longrightarrow \mathbb{R}$ be given by $g(x) = d(x, F_2)$ for $x \in F_1$. By $A.1.1$, g is a continuous map. Furthermore, F_1 is compact, and so $g(F_2)$ is a compact subset of $\mathbb{R}$. By the same argumemt as above, it must attain a minimum value at some $y \in F_1$. Let $\mu = g(y)$. We then have that $d(a_1, a_2) \geq \mu$ for $a_1 \in F_1, a_2 \in F_2$. It remains to show that $\mu > 0$. Consider $h : F_2 \longrightarrow \mathbb{R}$ given by $h(z) = d(y, z)$ for $x \in F_2$. h is continuous and F_2 is compact, so as before, h must attain a minimum at some $w \in F_2$, i.e., $\mu = d(w, y)$. Since F_1 and F_2 are disjoint, $y \neq w$ and so $\mu > 0$. □

A.3 **Contracting maps.** Let (E, d) be a compact metric space and let $f : E \to E$ be a continuous map such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in E$ with $x \neq y$. Define the diameter of a closed subset F of E to be $diam(F) = \sup_{x,y \in F} d(x, y)$.

(0) What are the closed subsets of E whose diameter is 0?

Proof. They are the closed subsets made of only one point. □

(1) Show that there exist $x, y \in F$ such that $diam(F) = d(x, y)$.

Proof. Since F is closed, it is compact in E and $F \times F$ is compact in $E \times E$ so d, which is a continuous function on $F \times F$, attains its maximum. Thus $\sup_{a,b\in F} d(a, b) = d(x, y)$ for some $x, y \in F$. $\square$

(2) Show that $diam(f(F)) < diam(F)$ if $diam(F) \neq 0$;

Proof. By A.3.1 applied to $f(F)$ which is also compact (being the continuous image of a compact set), we see that $diam(f(F)) = d(f(w), f(z))$ for some $w, z \in F$ and so $d(w, z) > diam(f(F)) = d(f(w), f(z))$. This means that there are two points in F further apart than $diam(f(F))$: more precisely, $diam(F) \geq d(w, z) > diam(f(F)) = d(f(w), f(z))$. $\square$

(3) Use this to show that f has a unique fixed point.

Proof. We will first prove the uniqueness of the fixed point in f. Suppose we have 2 distinct fixed points, x and y. Then $d(x, y) = d(f(x), f(y)) < d(x, y)$, but this is a contradiction.

Now, we prove existence: choose $x_0 \in E$ and define a sequence x_n by $x_{n+1} = f(x_n)$. We will prove that x_n converges to the fixed point of f. Let A be the set of accumulation points of x_n. We've seen in the previous homework that A is closed. Furthermore, $f(A) \subset A$ because if $x_{\phi(n)} \to a \in A$ then $x_{\phi(n)+1} = f(x_{\phi(n)}) \to f(a) \in A$. Finally $f : A \to A$ is surjective: if $x_{\phi(n)} \to a \in A$ then the sequence $x_{\phi(n)-1}$ has an accumulation point b and it's not hard to see that $b \in A$ and that $f(b) = a$. Therefore $f(A) = A$ which contradicts (2) unless A is a point. This implies that x_n has a unique accumulation point, i.e. a limit, which is easily seen to be the fixed point of f. $\square$

A.4 **Accumulation points II.** Let E be a compact metric space, let $u = \{u_n\}_n$ be a sequence of elements of E, and let $A(u)$ be the set of its accumulation points. Show that if $d(u_n, u_{n+1}) \to 0$ as $n \to +\infty$ then $A(u)$ is connected.

Hint: if you can write a space X as the disjoint union of two open sets, then those open sets are also closed.

Proof. We prove this by contradiction. Suppose $A(u)$ is the disjoint union of two open sets F_1, F_2. Then F_1, F_2 are also closed because F_1 is the complement of the open set F_2 and vice-versa. Then F_1 and F_2 are also compact, being closed in E.

By A.2.2, there exists $\mu > 0$ such that for $a_1 \in F_1, a_2 \in F_2, d(a_1, a_2) \geq \mu$. If $d(u_n, u_{n+1}) \to 0$ as $n \to +\infty$ then there exists N such that if $n \geq N$ then $d(u_n, u_{n+1}) < \mu$. This implies that all the u_n's with $n \geq N$ are in F_1 or that all of them are in F_2, and this in turn implies that $A(u) = F_1$ or that $A(u) = F_2$. Hence one of F_1, F_2 is necessarily empty and so $A(u)$ is connected. $\square$