Solution Set 2: Part B

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B.1. (1) *Show that $(\mathbb{Q}, d_p)$ is a metric space and that $d(x,z) \leq \max(d(x,y), d(y,z))$.*

Solution. We first have to verify the axioms for a metric, namely that $\forall x, y, z \in \mathbb{Q}$

- $d(x,y) \geq 0$
- $d(x,y) = 0 \Leftrightarrow x = y$
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(y,z)$

$d(x,y)$ is defined as either a power of p or 0, and therefore satisfies the first condition. For any x, y, $v(x-y) = v(y-x)$, since we can pull exactly the same number of factors out of $x-y$ and $y-x$, so $d(x,y) = d(y,x)$. $d(x,y)$ is a power of p if $x \neq y$, and zero otherwise, so it is nonnegative and zero if and only if $x = y$. Suppose $x - y = p^{v_1}\frac{n_1}{d_1}$, $z - y = p^{v_2}\frac{n_2}{d_2}$, with n_i, d_i relatively prime to p. We can assume $v_1 \leq v_2$, so that $v(x-z) = v((x-y)-(z-y)) \geq v_1$, for

$$x - z = p^{v_1}\frac{n_1}{d_1} - p^{v_2}\frac{n_2}{d_2} = p^{v_1}\left(\frac{n_1 d_2 - n_2 d_1 p^{v_2 - v_1}}{d_1 d_2}\right)$$

and since d_i are relatively prime to p, we can't pull out any more factors of p from the denominator, although we may be able to factor some out of the numerator (if $v_1 = v_2$). Thus,

$$d(x,z) = p^{-v(x-z)} \leq p^{-v_1} = \max(p^{-v_1}, p^{-v_2}) = \max(d(x,y), d(y,z))$$

The triangle inequality, $d(x,z) \leq d(x,y) + d(y,z)$, then follows. □

(2) *Show every element of $B(a,r)$ is a center of $B(a,r)$*

Solution. Note that every ball $B(a,r)$ is nonempty since $a \in B(a,r)$. For any $x \in B(a,r)$, $d(x,a) < r$. Pick any $y \in B(x,r)$, so that $d(x,y) < r$. By the above, $d(y,a) \leq \max(d(x,a), d(x,y)) = r$, so $y \in B(a,r)$, and therefore $B(x,r) \subset B(a,r)$. Certainly $a \in B(x,r)$, so applying the same reasoning $B(a,r) \subset B(x,r) \Rightarrow B(x,r) = B(a,r)$. □

(3) *Show that any given two balls are either disjoint or one is contained in the other*

Solution. Suppose there is an element $x \in B(a_1, r_1) \cap B(a_2, r_2)$. From part (2), $B(x, r_1) = B(a_1, r_1)$ and $B(x, r_2) = B(a_2, r_2)$. Without loss of generality we may assume $r_1 \leq r_2$, so $B(a_1, r_1) = B(x, r_1) \subset B(x, r_2) = B(a_2, r_2)$. Thus, either the two balls intersect, and therefore one is contained in the other, or they are disjoint. □

B.2. *If $u = \{u_n\}_{n \geq 1}$ is a sequence of elements in a metric space (E, d), show that the set $A(u)$ of accumulation points is closed.*

Solution. To show that $A(u)$ is closed, we will prove that the complement of $A(u)$ is open. Take any point $x \notin A(u)$. x is not an accumulation point of $\{u_n\}$, so for some ϵ there can be no points of the sequence closer to x than ϵ, i.e., $B(x,\epsilon) \cap \{u_n\} = \varnothing$; otherwise, x would be an accumulation point. There can be no accumulation point $u \in B(x,\epsilon)$. To see this, suppose there was, and let $\delta = d(x,u)$. $B(u, \epsilon - \delta)$ is contained in $B(x, \epsilon)$, since if y is in the former, $d(y,u) < \epsilon - \delta \Rightarrow d(x,y) \le d(y,u) + \delta < \epsilon$. But $B(u, \epsilon - \delta)$ doesn't contain any elements of the sequence $\{u_n\}$, so u is not an accumulation point. We have therefore produced a ball around an arbitrary $x \in X \setminus A(u)$ that is contained in $X \setminus A(u)$, and thus $X \setminus A(u)$ is open.

. Alternatively, we can show that for any sequence x_n **in $\mathbf{A(u)}$**, if x_n converges to x then $x \in A(u)$. For every ball $B(x, \frac{1}{2m})$, there exists an N such that $x_n \in B(x, \frac{1}{2m})$ for $n > N$. x_{N+1} is an accumulation point, and therefore there must be an element of u, say u'_m, that is in $B(x_{N+1}, \frac{1}{2m})$. We can pick the u'_m to be different from our previous choices since there are always infinitely many elements of the sequence in $B(x_{N+1}, \frac{1}{2m})$, and therefore the u'_m may be rearranged into a subsequence of u_n. For any m, $u'_m \in B(x, \frac{1}{m})$ by the triangle inequality, and since $\frac{1}{m}$ converges to 0, u'_m converges to x. Hence, $x \in A(u)$, and $A(u)$ is closed. □

B.3. *Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. Show that f is uniformly continuous.*

Solution. If we know $f(x) = \sqrt{x}$ is continuous, then this problem is really easy. We proved in class that every continuous function on a compact space is uniformly continuous, and that $[0,1]$ is compact, so f must be uniformly continuous.

. To make this problem more fun, we will now show that f is continuous. Let $x_0 \in [0,1]$, and let $\epsilon > 0$ be given. If we take $\delta = \epsilon^2$, then by the triangle inequality

$$|\sqrt{x} - \sqrt{x'}| \le \sqrt{x} + \sqrt{x'}$$

and therefore

$$|x - x'| = |\sqrt{x} - \sqrt{x'}|(\sqrt{x} + \sqrt{x'}) < \delta = \epsilon^2 \Rightarrow |\sqrt{x} - \sqrt{x'}|^2 \le |x - x'| < \epsilon^2 \Rightarrow |\sqrt{x} - \sqrt{x'}| < \epsilon$$

□

B.4. *Check that a K-Lipshitz function is uniformly continuous. Show that for every $0 < a < 1$, there is a K_a such that $f(x) = \sqrt{x}$ is K_a-Lipshitz on $[a,1]$. What happens at $a = 0$?*

Solution. Given that $f : X \to Y$ is K-Lipshitz, then for all $\epsilon > 0$, take $\delta = \epsilon/K$. For any $x, x' \in X$, we then have

$$d_X(x,x') < \delta = \epsilon/K \Rightarrow d_Y(f(x), f(x')) \le K d_Y(x,x') < \epsilon$$

and f is uniformly continuous.

. For $0 < a < 1$, take $K_a = \frac{1}{2\sqrt{a}}$ (we might guess this because we can think of K_a as a bound on the derivative—see below—but we shouldn't use this intuition in the formal proof!). For all $x, x' \in [a,1]$, $a \le x \le 1$ so that $\sqrt{a} \le \sqrt{x}$, and similarly for x'. Thus, $1/K_a \le \sqrt{x} + \sqrt{x'} \Rightarrow 1 \le K_a(\sqrt{x} + \sqrt{y})$, and

$$|\sqrt{x} - \sqrt{x'}| \le K_a|\sqrt{x} - \sqrt{x'}|(\sqrt{x} + \sqrt{x'}) \le K_a|x - x'|$$

The above does not work if $a = 0$, for then K_a is not defined. Heuristically, you can think of the constant K as a bound on the "derivative,"

$$\frac{|f(x) - f(x')|}{|x - x'|} \le K$$

But the derivative of $\sqrt{x}$, which is $-x^{-1/2}$, is unbounded at the origin. □